Ordinary Differential Equations (ODEs)

Differential Equation:
A differential equation is, in simpler terms, a statement of equality having a derivative or differentials.
An equation involving differentials or differential co-efficient is called a differential equation.

For examples, $\frac{d^2y}{dx^2}=0$ and $y\,dx+x\,dy=0$ are two differential equations.

Ordinary Differential Equation:
If a differential equation contains one dependent variable and one independent variable, then the differential equation is called ordinary differential equation.

For example, (i) $\frac{dy}{dx}=x\,Sinx$;

(ii) $4\frac{d^2y}{dx^2}+6y=\tan x$.

Partial Differential Equation:
If there are two or more independent variables, so that the derivatives are partial, then the differential equation is called partial differential equation.

For example, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$.

Order:
By the order of a differential equation, we mean the order of the highest differential co-efficient which appears in it.

For example, $4\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$ is a second order differential equation.

Degree:
By the degree of a differential equation, we mean the degree of the highest differential co-efficient after the equation has been put in the form free from radicals and fraction.

For example, $\left(\frac{d^2y}{dx^2}\right)^4+2x\left(\frac{dy}{dx}\right)^5=0$ is a differential equation whose degree is 4.

The degree of $\sqrt[4]{\frac{dy}{dx}+2x\left(\frac{d^4y}{dx^4}\right)^3}=\sqrt[3]{x-2}$ is 9.

$$\left\{\frac{dy}{dx}+2x\left(\frac{d^4y}{dx^4}\right)^3\right\}^{1/4}=(x-2)^{1/3}$$

$$\left\{\frac{dy}{dx} + 2x\left(\frac{d^4y}{dx^4}\right)^3\right\}^3 = (x-2)^4.$$

General Solution:
The solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the general solution.

For example, $y = ax+b$ is the general solution of the differential equation $\frac{d^2y}{dx^2}=0$,

where a and b are arbitrary constant.

Particular Solution:
If particular values are given to the arbitrary constants in the general solution, then the solution so obtained is called particular solution.

For example, putting $a=2$ and $b=3$ a particular solution of $\frac{d^2y}{dx^2}=0$ is $y = 2x+3$.

How to solve $\frac{d^2y}{dx^2}=0$ **?**

<u>**Solution**</u>:

$\frac{d^2y}{dx^2}=0$, or $\frac{d}{dx}\left(\frac{dy}{dx}\right)=0,$ or $\int d\left(\frac{dy}{dx}\right)=a,$

or $\int du=a$ where $u=\frac{dy}{dx},$

or $u=a$,

or $\frac{dy}{dx}=a$,

or $\int dy=a\int dx+b,$

or $y=ax+b.$

which geometrically represents a straight line.

Or one may solve it in the following way :

$\frac{d^2y}{dx^2}=0$, or $\frac{d}{dx}\left(\frac{dy}{dx}\right)=0;$

since the derivative of $\frac{dy}{dx}$ is zero;

so $\frac{dy}{dx}=$ constant a (say)

or $\int dy=a\int dx+b$ or $y=aqx+b.$

In fact $\frac{d^2y}{dx^2}=0$ is an ODE of order 2 and has as solution with 2 parameters a and b.

Formation of Ordinary Differential Equation:

Eliminating arbitrary constants, we can form ODE.

Form an ODE corresponding to $y = e^x (A\,Cosx + B\,Sinx)$

Solution:

Given, $y = e^x (A\,Cosx + B\,Sinx)$

Differentiating with respect to x we get

$$\frac{dy}{dx} = e^x (A\,Cosx + B\,Sinx) + e^x (-A Sinx + B\,Cosx)$$

$$= y + e^x (-A Sinx + B\,Cosx)$$

Differentiating again with respect to x we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x (-A Sinx + B\,Cos\,x) + e^x (-A\,Cosx - B\,Sinx)$$

$$= \frac{dy}{dx} + e^x (-A Sinx + B\,Cos\,x) - e^x (A\,Cosx + B\,Sinx)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

Derive an ODE corresponding to all circles lying in a plane.

Derivation:

The equation of all circles lying in a plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(i)}$$

Which contains three arbitrary constant g, f and c.

Differentiating (i) thrice successively, we have

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + g + f\frac{dy}{dx} = 0$$

$$1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} + f\frac{d^2y}{dx^2} = 0$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y + f)\frac{d^2y}{dx^2} = 0$$

$$1 + \left(\frac{dy}{dx}\right)^2 = -(y + f)\frac{d^2y}{dx^2} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(ii)}$$

$$2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = -(y+f)\frac{d^3y}{dx^3} - \frac{dy}{dx}\frac{d^2y}{dx^2}$$

$$3\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = -(y+f)\frac{d^3y}{dx^3} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(iii)}$$

Dividing (ii) by (iii)

$$\frac{1+\left(\frac{dy}{dx}\right)^2}{3\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}} = \frac{-(y+f)\frac{d^2y}{dx^2}}{-(y+f)\frac{d^3y}{dx^3}}$$

$$3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2 = \frac{d^3y}{dx^3}\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

Find the differential equation of the curves $xy = Ae^{2x} + Be^{-2x}$ for different values of A and B. State order and degree of the derived equation.

Solution:

Given equation, $xy = Ae^{2x} + Be^{-2x}$

Differentiating with respect to x we get

$$x\frac{dy}{dx} + y = 2Ae^{2x} - 2Be^{-2x}$$

Differentiating again with respect to x we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 4Ae^{2x} + 4Be^{-2x}$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4\left(Ae^{2x} + Be^{-2x}\right)$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} - 4xy = 0$$

This is the required differential equation. The order of the above equation is 2 and the degree is 1.

Show that $Ax^2 + By^2 = 1$ is the solution of $x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$.

Proof:

Given equation, $Ax^2 + By^2 = 1$

Differentiating with respect to x we get

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$Ax + By\frac{dy}{dx} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(i)}$$

$$\frac{A}{B} = -\frac{y}{x}\frac{dy}{dx} \quad \dots \dots \dots \dots \dots \dots \dots \dots \text{(ii)}$$

Differentiating (i) again with respect to x we get

$$A + By\frac{d^2y}{dx^2} + B\left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{A}{B} = -y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2$$

$$-\frac{y}{x}\frac{dy}{dx} = -y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 \qquad \text{[Using (ii)]}$$

$$x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}.$$

Find the differential equations of all circles passing through origin and having their centres on the X-axis.

Solution:

The equations of all circles passing through origin and having their centres on the X-axis is

$$x^2 + y^2 + 2gx = 0 \quad \dots \dots \dots \dots \dots \dots \dots \dots \text{(i)}$$

$$2gx = -\left(x^2 + y^2\right)$$

$$2g = -\left(\frac{x^2 + y^2}{x}\right) \quad \dots \dots \dots \dots \dots \dots \dots \dots \text{(ii)}$$

Differentiating (i) with respect to x we get

$$2x + 2y\frac{dy}{dx} + 2g = 0$$

$$2x + 2y\frac{dy}{dx} - \left(\frac{x^2 + y^2}{x}\right) = 0.$$