Session 01
Course Title: ODE & PDE
Course Code: MAT211

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Topics to be covered

Introduction to Differential Equations (DE), order and degree of DE, solution types, ODE & PDE, Linear and Nonlinear DE, Formation of DE.

Expected Outcomes

- Basics on DE and able to find order and degree of DE,
- Identify linear and nonlinear DE
- Learn about ODE & PDE
- Basics on general solution and Particular solution

They will able to form a DE from its primitive

Introduction to Differential Equations (D.Es.)

Differential Equation:

A differential equation is, in simpler terms, a statement of equality having a derivative or differentials. An equation involving differentials or differential co-efficient is called a differential equation.

For Example, $\frac{d^2y}{dx^2} = 0$ and $ydx + xdy = 0$ are two differential equations

For Example, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

Differential Equations

Ordinary D.E (O.D.E.):

$$(i)\ \frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$(ii)\ 2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = e^x + \sin x$$

Partial D.E (P.D.E.):

$$(i)\ x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial y^2} = \tan x$$

$$(ii)\ \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

Ordinary Differential Equation (ODE):

If a differential equation contains one/more dependent variable and one independent variable, then the differential equation is called ordinary differential equation.

Partial Differential Equation(PDE):

If there are two or more independent variables, so that the derivatives are partial, then the differential equation is called partial differential equation.

Ordinary D.E (O.D.E.):

$$(i)\ \frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$(ii)\ 2\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = e^x + \sin x$$

Partial D.E (P.D.E.):

$$(i)\ x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial y^2} = \tan x$$

$$(ii)\ \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

Order:
By the order of a differential equation, we mean the order of the highest differential coefficient which appears in it.

Degree:
By the degree of a differential equation, we mean the degree of the highest differential coefficient after the equation has been put in the form free from radicals and fraction.

For Example, $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0$ is a second order differential equation

For Example, $\left(\frac{d^2y}{dx^2}\right)^4 + 5x\left(\frac{dy}{dx}\right)^5 = 0$ is a differential equation of degree 4 because we count degree based on highest order in a differential equation.

$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -4\cos 2x \quad \rightarrow$ D.E. with second order first degree

Note:

- Degree of a differential equation is defined if it is a polynomial equation in its derivative.
- The degree of the differential equation is always positive, but never a negative or zero or fraction.
- Dependent variable should not include fraction powers, it should be perfectly linear. For example

$\frac{d^2y}{dx^2} + \sqrt{y} = 0$, degree does not exist

- Degree of the DE does not exist when the differential coefficient involving with exponential functions, logarithmic functions and trigonometric functions. For example:

- There is no degree for DE $e^{\frac{dy}{dx}} + 1 = 0$
- There is no degree for DE $\ln\left(\frac{dy}{dx}\right) + 1 = 0$
- There is no degree for DE $\sin\left(\frac{dy}{dx}\right) + 1 = 0$

Order and Degree of D.E.

$(i)\ \frac{dy}{dx}=\frac{1+y^2}{1+x^2}\ \rightarrow$ first order and first degree

$(ii)\ 2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+2y=e^x+\sin x\ \rightarrow$ second order and first degree

$(iii)\ x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y^2}=\tan x\ \rightarrow$ second order and first degree

$(iv)\ \frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}=0\ \rightarrow$ first order and first degree

Problem: Find the order and degree of the differential equation $\frac{d^3y}{dx^3} = \sqrt[5]{\left(\frac{dy}{dx}\right)^2 + 5\frac{dy}{dx} + y}$

Solution:

$$\frac{d^3y}{dx^3} = \sqrt[5]{\left(\frac{dy}{dx}\right)^2 + 5\frac{dy}{dx} + y}$$

Rationalize the above equation, we get

$$\left(\frac{d^3y}{dx^3}\right)^5 = \left(\frac{dy}{dx}\right)^2 + 5\frac{dy}{dx} + y$$

Here highest derivative is 3, so the order is of the DE is 3

The power of the highest order derivatives is 5, so the degree of this DE is 5

Classification based on Linearity:

Linear ordinary differential equation:

An ordinary differential equation of order n is called a linear ordinary differential equation of order n if it follow the followings conditions

1. No transcendental functions of dependent variable or its derivative exists

2. No product of dependent variables and its derivatives

3. The dependent variable and all its derivatives are of the first degree i.e. the power of each term involving is y is 1.

Transcendental function: In mathematics, a function not expressible as a finite combination of the algebraic operations of addition, subtraction, multiplication, division, raising to a power, and extracting a root. Examples include the functions $log\ x, sin\ x, cos\ x, e^x$ and any functions containing them. Such functions are expressible in algebraic terms only as infinite series.

It can be expressed as

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

Where a_0 is not identically zero.

For example:

$\frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = 0$; linear third order ordinary differential equation

Nonlinear ordinary differential equation: A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

Example:

1. $\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0$
2. $\frac{d^3 y}{dx^3} + e^y\frac{d^2 y}{dx^2} + xy = xe^x$
3. $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 6y = 0$

Solution of Differential Equation

General Solution:
The solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the general solution.

Particular Solution:
If particular values are given to the arbitrary constants in the general solution, then the solution so obtained is called particular solution.

Example: $y = ax + b$
Differentiating w. r. to x

$$\frac{dy}{dx} = a.1 + 0$$

derivative again w. r. to x

$$\frac{d^2y}{dx^2} = 0$$

So, $y = ax + b$ is the general solution of the differential equation $\frac{d^2y}{dx^2} = 0$, where a and b are arbitrary constant

For Example,Putting a=2 and b=3, a particular solution of $\frac{d^2y}{dx^2} = 0$ is y=2x+3

Formation of Ordinary Differential Equation(ODE) by eliminating arbitrary constants

Example : Form an ODE of $y = e^{x}(ACosx + BSinx)$

$$\frac{d}{dx}(uv) = uv' + vu'$$

Solution: Differentiating the above with respect to x

$$\frac{d}{dx}e^{x} = e^{x}$$

$$\frac{dy}{dx} = e^{x}\frac{d}{dx}(ACosx + BSinx) + (ACosx + BSinx)\frac{d}{dx}(e^{x})$$

$$\frac{d}{dx}(sinx) = cosx$$

$$\frac{dy}{dx} = e^{x}(-ASinx + BCosx) + e^{x}(ACosx + BSinx)$$

$$\frac{d}{dx}(cosx) = -sinx$$

$\frac{dy}{dx} = e^{x}(-ASinx + BCosx) + y$, Since $y = e^{x}(ACosx + BSinx)$.........(*)

$$\frac{d^2y}{dx^2} = e^{x}\frac{d}{dx}(-ASinx + BCosx) + (-ASinx + BCosx)\frac{d}{dx}(e^{x}) + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = e^{x}(-ACosx - BSinx) + e^{x}(-ASinx + BCosx) + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -e^x(ACosx + BSinx) + e^x(-ASinx + BCosx) + \frac{dy}{dx}$$

$\frac{d^2y}{dx^2} = -y + \frac{dy}{dx} - y + \frac{dy}{dx}$, Since $e^x(-ASinx + BCosx) = \frac{dy}{dx} - y$ from equation (*)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Since there is no arbitrary constant so this is the required ODE. We differentiated two times because of having 2 arbitrary constants initially.

Example: Form an ODE of $xy = Ae^{2x} + Be^{-2x}$

$$\frac{d}{dx}e^{mx} = me^{mx}$$

Solution: Differentiating the above with respect to x we have,

$$\frac{d}{dx}(uv) = uv' + vu'$$

$x\frac{dy}{dx} + y = 2Ae^{2x} - 2Be^{-2x}$ using the calculus formula of $\frac{d}{dx}(uv)$

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 4Ae^{2x} + 4Be^{-2x}$, again differentiating w.r.t x

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4(Ae^{2x} + Be^{-2x})$$

$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4xy$, Since $xy = Ae^{2x} + Be^{-2x}$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4xy=0$$

Since there is no arbitrary constant so this is the required ODE. We differentiated two times because of having 2 arbitrary constants initially. Finally, the order of this equation is 2 and the degree is 1

Formation of D.E.

Problem :

Form the D.E. corresponding to the equations

$(a)\ \ y = ax + bx^2$ $\qquad (b)\ \ c(y+c)^2 = x^3$

$(c)\ \ y = ae^{2x} + be^{-3x} + ce^{x}$ $\qquad (d)\ \ y = cx + c - c^3$

$(e)\ \ e^{2y} + 2cxe^{y} + c^2 = 0$ $\qquad (f)\ \ xy = ae^{x} + be^{-x}$

$(g)\ \ xy = Ae^{x} + Be^{-x} + x^2$

Formation of D.E.

Problem : Form the D.E. corresponding the equation $y = ax + bx^2$

Solution:

Given that

$$y = ax + bx^2 \qquad (1)$$

Differenting both sides w. r. to x, we get

$$\frac{dy}{dx} = a + 2bx \qquad (2)$$

Again differenting both sides w. r. to x

$$\frac{d^2y}{dx^2} = 2b \quad \text{Þ} \quad b = \frac{1}{2}\frac{d^2y}{dx^2}$$

Putting the value of b in equation (2), we get

$$\frac{dy}{dx} = a + 2.\frac{1}{2}\frac{d^2y}{dx^2}x = a + \frac{d^2y}{dx^2}x$$

$$\text{Þ} \quad \mathrm{a} = \frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

Formation of D.E.

Putting the values of a and b in equation (1), we get

$$y = \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2} \right)x + \frac{1}{2}\frac{d^2y}{dx^2}x^2$$

$$= x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2} + \frac{1}{2}\frac{d^2y}{dx^2}x^2$$

$$= x\frac{dy}{dx} - \frac{1}{2}\frac{d^2y}{dx^2}x^2$$

$$\Rightarrow 2y = 2x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

which is a D.E. of second order and first degree.

Formation of D.E.

Solution (b): Given that

$$c(y+c)^2 = x^3 \qquad (1)$$

Differenting both sides w. r. to x we get,

$$2c(y+c).\frac{dy}{dx} = 3x^2 \qquad (2)$$

Dividing (1) by (2), we get

$$\frac{y+c}{2\,dy/dx} = \frac{x}{3}$$

$$\text{Þ}\ 3(y+c) = 2x\frac{dy}{dx}\ \text{Þ}\ y+c = \frac{2}{3}x\frac{dy}{dx}$$

$$\text{Þ}\ c = \frac{2}{3}x\frac{dy}{dx} - y$$

Now putting the value of c in equation (2), we get

$$\left(2.\frac{2}{3}x\frac{dy}{dx} - 2y\right)\left(y + \frac{2}{3}x\frac{dy}{dx} - y\right).\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{4}{3}x\frac{dy}{dx} - 2y\right).\frac{2}{3}x\frac{dy}{dx}.\frac{dy}{dx} = 3x^2$$

Formation of D.E.

$$\Rightarrow \frac{8}{9}x^2\left(\frac{dy}{dx}\right)^3 - \frac{4}{3}xy\left(\frac{dy}{dx}\right)^2 = 3x^2$$

$$\Rightarrow 8x\left(\frac{dy}{dx}\right)^3 - 12y\left(\frac{dy}{dx}\right)^2 - 27x = 0$$

which is a D.E. of first order and 3rd degree.

Formation of D.E.

Solution (c): Given that $y = ae^{2x} + be^{-3x} + ce^{x}$ (1)

Differenting both sides w. r. to x, we get

$$\frac{dy}{dx} = 2ae^{2x} - 3be^{-3x} + ce^{x} \qquad (2)$$

Again differenting w. r. to x, we get

$$\frac{d^2y}{dx^2} = 4ae^{2x} + 9be^{-3x} + ce^{x} \qquad (3)$$

Again differenting w. r. to x

$$\frac{d^3y}{dx^3} = 8ae^{2x} - 27be^{-3x} + ce^{x}$$

Þ $\frac{d^3y}{dx^3} = 14ae^{2x} - 21be^{-3x} + 7ce^{x} - 6ae^{2x} - 6be^{-3x} - 6ce^{x}$

Þ $\frac{d^3y}{dx^3} = 7(2ae^{2x} - 3be^{-3x} + ce^{x}) - 6(ae^{2x} + be^{-3x} + ce^{x})$

Þ $\frac{d^3y}{dx^3} = 7\frac{dy}{dx} - 6y$ [using (2) and (1)]

Þ $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$

Exercise

1. Show that the differential equation of a family of circles touches the $x - axis$ at origin is $(x^2 - y^2)dy - 2xydx = 0.$

2. Form the differential of parabolas $y^2 = 4a(x + a)$. $[Ans. \rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.]$

3. Form the differential equation from the curve $y = Ae^{2x} + Be^{-2x}$. $[Ans. \rightarrow \frac{d^2y}{dx^2} - 4y = 0.]$

4. Form a differential equation of $y = ax + \frac{b}{x}$ $[Ans. \rightarrow x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 0.]$

5. Form the differential equation from the curve $r = a + bcos\theta$. $[Ans. \rightarrow \frac{d^2r}{d\theta^2} = cot\theta\frac{dr}{d\theta}.]$

6. Find the differential equation whose solution $y = e^x(Acosx + Bsinx)$. $[Ans. \rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.]$

7. Find the differential equation whose solution $Ax^2 + By^2 = 1$. $[Ans. \rightarrow x\left[y\frac{d^2y}{dx^2} +\right.$