

**Protima Dash**

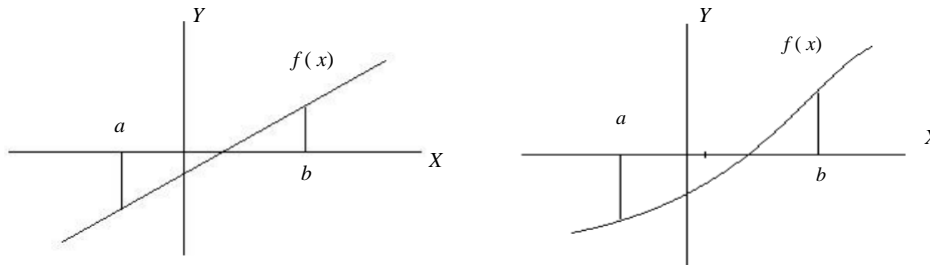
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## *Solution of Algebraic and Transcendental equations*

In this Lesson, we have discussed about the solution of equations  $f(x) = 0$  where  $f(x)$  linear, non-linear, algebraic or transcendental function. We get the solution of the equation  $f(x) = 0$  by using Bisection method, Newton- Raphson method and method of false position. Those methods are established based on **Intermediate Value Theorem**.

*Statement of Intermediate Value Theorem:*

*If  $f(x)$  is continuous in the interval  $(a, b)$  and if  $f(a)$  and  $f(b)$  are of opposite signs, then the equation  $f(x) = 0$  will have at least one real root between  $a$  and  $b$ .*



### Algebraic equation:

An algebraic equation is an equation that includes one or more variables such as  $x^2 + xy - z = 0$ .

### Transcendental equation:

An equation together with algebraic, trigonometrical, exponential or logarithmic function etc. is called transcendental equation such as  $e^x + 2\sin x - 5x = 0$ .

### Solution/root:

A solution/root of an equation is the value of the variable or variables that satisfies the equation.

### Iteration:

Iteration is the repeated process of calculation until the desired result or approximate numerical value has come. Each repetition of the process is also called iteration and the result of one iteration is used as the starting point for the next iteration.

We are capable to find the root of *algebraic or transcendental function by using following methods:*

1. Bisection method
2. Newton Rapshon method (Newton's Iteration method)
3. Iteration method (Method of successive approximation/Fixed-point Iteration Method)
4. Regular-Falsi method (The method of False position)
5. The secant method
6. Muller's method
7. Ramanujan's method
8. Horner's method

## Bisection Method:

Let us suppose we have an equation of the form  $f(x) = 0$  in which solution lies between in the range  $(a, b)$  where  $a < b$ . Also  $f(x)$  is continuous and it can be algebraic or transcendental. If  $f(a)$  and  $f(b)$  are opposite signs, then there exist at least one real root between  $a$  and  $b$ . Let  $f(a)$  be positive and  $f(b)$  negative. Which implies at least one root exists between  $a$  and  $b$ . We assume that root to be

$x_0 = \frac{a+b}{2}$ . Check the sign of  $f(x_0)$ . If  $f(x_0)$  is negative, the root lies between  $a$  and  $x_0$ . If  $f(x_0)$  is

positive, the root lies between  $x_0$  and  $b$ .

Subsequently any one of this case occur.  $x_1 = \frac{a+x_0}{2}$  Or  $x_1 = \frac{b+x_0}{2}$ . When

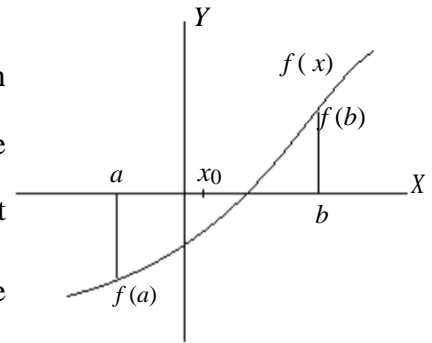
$f(x_1)$  is negative, the root lies between  $x_0$  and  $x_1$  and let the root be

$x_2 = \frac{x_1+x_0}{2}$ . Again  $f(x_2)$  negative then the root lies between  $x_0$  and  $x_2$ , let

$x_3 = \frac{x_0+x_2}{2}$  and so on. Repeat the process  $x_0, x_1, x_2, \dots, x_{k-1}, x_k$  whose

limit of convergence is the exact root. We have to stop the iteration when the value of two successive iterations are approximately equal. That is

$$x_{k-1} \approx x_k \quad \text{or} \quad |x_k - x_{k-1}| \approx 0$$



### Advantages of the bisection method:

1. It is always convergent.
2. The error bound decreases by half with each iteration i.e., error can be controlled.
3. It is well suited to electronic Computers.
4. It is very simple method.

### Disadvantages/draw-back of the bisection method:

1. The bisection method converges very slowly
2. It requires large number of iterations
3. The bisection method cannot detect multiple roots
4. Choosing a guess close to the root may result in needing many iterations to converge.
5. Cannot find roots of some equations such as  $y = x^2 = 0$  because upper guess and lower guess always produce positive value.
6. May seek a singularity point as a root as the equation like  $y = \frac{1}{x} = 0$

**Algorithm for Bisection method:**

| Steps | Task  |
|-------|---|
| 01    | Define $f(x)$   |
| 2     | Read a 'The lower bound of the desired roots'   |
| 3     | Read b 'The upper bound of the desired roots'   |
| 4     | Set $k = 1$   |
| 05    | Calculate $x_k = \frac{a + b}{2}$   |
| 06    | Calculate $f_k = f(x_k)$  |
| 07    | Print $k, x_k, f_k$   |
| 8     | If $ x_k - x_{k-1}  \approx 0.0001$ then<br>GOTO Step 11<br>elseif<br>$f(a) \cdot f_k < 0$ then $b = x_k$ .<br>Else<br>$f(b) \cdot f_k < 0$ then $a = x_k$ .<br>Endif |
| 9     | Set $k = k + 1$   |
| 10    | GOTO Step 05  |
| 11    | Print 'Required root, $x_k$ '   |
| 12    | STOP  |

**Problem 01:**

Find a root of the equation  $x^2 - 4x - 10 = 0$  using Bisection method.

**Solution:**

Let  $f(x) = x^2 - 4x - 10$

Here, let  $a = -2$  then  $b = -1$

$f(-2) = 4 + 8 - 10 = 2 > 0$  and  $f(-1) = 1 + 4 - 10 = -5 < 0$  .

Since  $f(-2)$  is positive and  $f(-1)$  is negative so at least one real root lies between -2 and -1.

$$\therefore x = \frac{-2 + (-1)}{2} = \frac{-3}{2} = -1.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

| Iteration | Value of a (+) | Value of b (-) | $x = \frac{a + b}{2}$ | Sign of $f(x) = x^2 - 4x - 10$ |
|-----------|----------------|----------------|-----------------------|--------------------------------|
| 1         | -2             | -1             | -1.5                  | -1.75 < 0                      |
| 2         | -2             | -1.5           | -1.75                 | 0.0625 > 0                     |

|    |         |         |         |             |
|----|---------|---------|---------|-------------|
| 3  | -1.75   | -1.5    | -1.625  | -0.859 < 0  |
| 4  | -1.75   | -1.625  | -1.6875 | -0.40 < 0   |
| 5  | -1.75   | -1.6875 | -1.7188 | -0.1705 < 0 |
| 6  | -1.75   | -1.7188 | -1.7344 | -0.054 < 0  |
| 7  | -1.75   | -1.7344 | -1.7422 | 0.004 > 0   |
| 8  | -1.7422 | -1.7344 | -1.7383 | -0.025 < 0  |
| 9  | -1.7422 | -1.7383 | -1.7402 | -0.0109 < 0 |
| 10 | -1.7422 | -1.7402 | -1.7412 | -0.003 < 0  |

The approximate root of the given equation is  $x = -1.7412$  because  $f(-1.7412) = -0.003 \approx 0$ .

**Problem 02:**

Find the root of the equation  $x^3 - x - 1 = 0$  by using Bisection method correct up to two decimal places.

**Solution:**

Let  $f(x) = x^3 - x - 1$

Here, let  $x = 1$  then  $x = 2$

$f(1) = 1 - 1 - 1 = -1 < 0$  and  $f(2) = 8 - 2 - 1 = 5 > 0$

Since  $f(1)$  and  $f(2)$  are of opposite sign so at least one real root lies between 1 and 2.

$$\therefore x = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

| Iteration | Value of a (+) | Value of b (-) | $x = \frac{a+b}{2}$ | Sign of $f(x) = x^3 - x - 1$ |
|-----------|----------------|----------------|---------------------|------------------------------|
| 1.        | 2              | 1              | 1.5                 | 0.875 > 0                    |
| 2.        | 1.5            | 1              | 1.25                | -0.297 < 0                   |
| 3.        | 1.5            | 1.25           | 1.375               | 0.2246 > 0                   |
| 4.        | 1.375          | 1.25           | 1.3125              | -0.0515 < 0                  |
| 5.        | 1.375          | 1.3125         | 1.34375             | 0.08626 > 0                  |
| 6.        | 1.34375        | 1.3125         | 1.3281              | 0.018447 > 0                 |
| 7.        | 1.3281         | 1.3125         | 1.3203              | -0.019 < 0                   |

|     |        |        |        |              |
|-----|--------|--------|--------|--------------|
| 8.  | 1.3281 | 1.3203 | 1.3242 | -0.002 < 0   |
| 9.  | 1.3281 | 1.3242 | 1.3261 | 0.005970 > 0 |
| 10. | 1.3261 | 1.3242 | 1.3251 | 0.00162 < 0  |

It is evident that from the above table, the difference between two successive iterative values of  $x$  is  $1.3261 - 1.3251 = 0.001$  which the accuracy condition for the solution exact. So, the required root of the given equation up to the two decimal places is 1.32.

**Problem 03:**

Find the root of the equation  $x e^x = 1$  by using Bisection method correct up to three decimal places on the interval  $(0, 1)$ .

**Solution:**

Let  $f(x) = x e^x - 1$

Here, let  $a = 0$  then  $b = 1$

$f(0) = 0.e^0 - 1 = -1 < 0$  and  $f(1) = 1.e^1 - 1 = 1.7182 > 0$

Since  $f(0)$  and  $f(1)$  are of opposite sign so at least one real root lies between 0 and 1.

$$\therefore x = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

| Iteration | Value of a ( + ) | Value of b ( - ) | $x = \frac{a+b}{2}$ | Sign of $f(x) = x e^x - 1$ |
|-----------|------------------|------------------|---------------------|----------------------------|
| 1.        | 1                | 0                | 0.5                 | -0.1756 < 0                |
| 2.        | 1                | 0.5              | 0.75                | 0.5877 > 0                 |
| 3.        | 0.75             | 0.5              | 0.625               | 0.1676 > 0                 |
| 4.        | 0.625            | 0.5              | 0.5625              | -0.0127 < 0                |
| 5.        | 0.625            | 0.5625           | 0.59375             | 0.0751 > 0                 |
| 6.        | 0.59375          | 0.5625           | 0.578125            | 0.0306 > 0                 |
| 7.        | 0.578125         | 0.5625           | 0.5703125           | 0.00877 > 0                |
| 8.        | 0.5703125        | 0.5625           | 0.56640625          | -0.0023 < 0                |

|     |           |            |               |              |
|-----|-----------|------------|---------------|--------------|
| 9.  | 0.5703125 | 0.56640625 | 0.56835<br>94 | 0.00336 > 0  |
| 10. | 0.5683594 | 0.56640625 | 0.56738<br>28 | 0.000662 > 0 |

It is evident that from the above table, the difference between two successive iterative values of  $x$  is  $0.5683594 - 0.5673828 \approx 0.001$  which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is  $\approx 0.567$ .

**Problem 04:**

Find the root of the equation  $4\sin x - e^x = 0$  by using Bisection method correct up to four decimal places.

**Solution:**

Consider that,

Here,

Since  $f(0)$  and  $f(1)$  are of opposite sign so at least one real root lies between 0 and 1.

$$\therefore x = \frac{0 + 1}{2} = \frac{1}{2} = 0.5$$

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.

| Iteration | Value of a (+) | Value of b (-) | $x = \frac{a+b}{2}$ | Sign of $f(x)$ |
|-----------|----------------|----------------|---------------------|----------------|
| 1.        | 1              | 0              | 0.5                 | 0.268 > 0      |
| 2.        | 0.5            | 0              | 0.25                | -0.294 < 0     |
| 3.        | 0.5            | 0.25           | 0.375               | 0.0101 > 0     |
| 4.        | 0.375          | 0.25           | 0.3125              | -0.1371 < 0    |
| 5.        | 0.375          | 0.3125         | 0.34375             | -0.0621 < 0    |
| 6.        | 0.375          | 0.34375        | 0.359375            | -0.0256 < 0    |
| 7.        | 0.375          | 0.359375       | 0.3671875           | -0.0077 < 0    |
| 8.        | 0.375          | 0.3671875      | 0.3710937           | -0.00122 < 0   |



|     |          |           |          |              |
|-----|----------|-----------|----------|--------------|
| 9.  | 0.375    | 0.3710937 | 0.373046 | 0.00566 > 0  |
| 10. | 0.373046 | 0.3710937 | 0.372070 | -0.00344 < 0 |
| 11. | 0.373046 | 0.372070  | 0.372558 | 0.00455 > 0  |
| 12. | 0.372558 | 0.372070  | 0.372279 | 0.0039 > 0   |
| 13. | 0.372279 | 0.372070  | 0.372174 | 0.0036 > 0   |
| 14. | 0.372174 | 0.372070  | 0.372122 | 0.0036 > 0   |

It is evident that from the above table, the difference between two successive iterative values of  $x$  is  $0.372174 - 0.372122 \approx 0.0001$  which is the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is  $\approx 0.3721$ .

**Note:**

To determine the value of the trigonometrical function  $f(x)$ , we have to change our calculator in radian mode.

## Try yourself:

**TYPE01:**

To find the root of the following equations using Bisection method by your own choosing interval

1.  $2^x - 5x + 2 = 0$
2.  $e^{2x} - e^x - 2 = 0$
3.  $x^3 + x^2 - 1 = 0$
4.  $2x + \cos x - 3 = 0$
5.  $\cos x - \ln x = 0$
6.  $x^2 - 4x - 10 = 0$
7.  $2x = 1 + \sin x$
8.  $x^3 - 2x^2 - 4 = 0$
9.  $\sin^2 x = x^2 - 1$
10.  $x \sin x = 1$
11.  $\cos x - xe^x = 0$
12.  $4x = \tan x$
13.  $e^x \tan x = 1$

**TYPE02:**

1. Apply bisection method to find real root of  $x^3 - 3x + 1 = 0$  that lies in  $(0, 1)$ .
2. Find the real root of the equation  $x + \ln x - 2 = 0$  belonging to the interval  $(1, 2)$  using Bisection Method.
3. Find the real root of the equation  $x + \log_{10} x - 1.2 = 0$  belonging to the interval  $(2, 3)$  using Bisection Method.
4. Find the real root of the equation  $e^{-x^2} - \cos x = 0$  belonging to the interval  $(1, 3)$  using Bisection Method.
5. Find the real root of  $e^x + 4x^2 = 0$  that lies in  $(0, 1)$ .
6. Find the real root of the equation  $x^3 - 3x - 5 = 0$  belonging to the interval  $(2, 3)$  using Bisection Method.
7. Find the real root of the equation  $x^3 - x - 1 = 0$  belonging to the interval  $(2, 3)$  using Bisection Method.
8. Find the positive real root of the equation  $x^3 - 3x + 1.06 = 0$  by Bisection Method correct to four decimal places.
9. Find the positive real root of the equation  $x^4 + x^2 - 80 = 0$  by Bisection Method correct to three decimal places.
10. Compute a root of the equation  $e^x = x^2$  to an accuracy of  $10^{-5}$  using bisection method.
11. Compute one root of  $e^x - 3x = 0$  correct to two decimal places.
12. Solve the equation  $x - \exp\left(\frac{1}{x}\right) = 0$  by bisection method.

**TYPE03:**

1. Discuss the Bisection Method to find a real root of the equation  $f(x) = 0$  in the interval  $[a, b]$ .
2. Write down an Algorithm for Bisection Method.
3. Mention the Merits and demerits of Bisection Method.
4. What are the draw-backs of the Bisection Method?

## Fixed Point Iteration Method:

Let us consider an equation  $f(x) = 0$  whose roots are to be determined in the interval ( ). The equation  $f(x) = 0$  can be expressed as

$$x = \varphi(x) \dots\dots\dots (1)$$

$$| \quad ' \quad |$$

Let  $x_0$  is an initial solution or approximation for the equation  $f(x) = 0$  .we substitute the value of  $x_0$  in the right-hand side of the equation (1) and obtain a better approximation  $x_1$  given by the equation  $x_1 = \varphi(x_0)$  . Again, substituting  $x = x_1$  in the equation (1), we get next approximation as  $x_2 = \varphi(x_1)$  . Preceding in this way we can find the following successive approximations,

$$\begin{aligned} x_3 &= \varphi(x_2) \\ x_4 &= \varphi(x_3) \\ &\dots\dots\dots \\ x_n &= \varphi(x_{n-1}) \end{aligned}$$

Therefore the iterative formula for successive approximation method is,

$$x_n = \varphi(x_{n-1}), n = 1, 2, 3, 4, \dots \text{etc.}$$

Here  $x_n$  is the n-th approximation of the desired root of  $f(x) = 0$  .

We shall continue this iterative cycle until the values of two successive approximations are almost equal. This above mentioned method is known as **Iteration method. Or Method of successive approximation or Fixed point Iteration.**

### Algorithm for Iteration method:

| Steps | Task   |
|-------|--|
| 01    | Define $\varphi(x)$  |
| 02    | Read $x_0$   |
| 03    | Set $k = 1$  |
| 4     | $x_n = \varphi(x_{n-1})$   |
| 05    | If $ x_n - x_{n-1}  \approx 0.0001$ then<br>GOTO Step 6<br>else<br>$n = n + 1$<br>GOTO Step 04 |
| 6     | Print $x_n$ , the desired root   |
| 7     | STOP   |

**Problem 01:**

Find the real root of the equation  $x^3 + x^2 - 1 = 0$  on the interval  $[0, 1]$  with an accuracy of  $10^{-4}$ .

**Solution:**

Let  $f(x) = x^3 + x^2 - 1$

$\therefore f(0) = 0^3 + 0^2 - 1 < -1$  and  $f(1) = 1^3 + 1^2 - 1 = 1 > 1$

Since  $f(0)$  and  $f(1)$  are of opposite sign so at least one real root lies between 0 and 1.

The given equation can be expressed as  $x^3 + x^2 - 1 = 0$

$$\begin{aligned} x^2(x+1) &= 1 \\ x^2 &= \frac{1}{(x+1)} \\ x &= \sqrt{\frac{1}{(x+1)}} = \varphi(x) \text{ [ say]} \\ \therefore \varphi(x) &= \frac{1}{\sqrt{(x+1)}} \Rightarrow \varphi(x) = -\frac{1}{2} \frac{1}{(x+1)^{\frac{3}{2}}} \end{aligned}$$

Also,  $|\varphi'(x)| = \left| -\frac{1}{2} \frac{1}{(x+1)^{\frac{3}{2}}} \right| = \frac{1}{2} \left| \frac{1}{(x+1)^{\frac{3}{2}}} \right| < 1$  for  $x \in (0,1)$ .

Therefore, the iteration method is applicable for the given function.

Assume  $x_0 = 0.7$  is an initial solution or approximation for the equation  $f(x) = 0$ .

So successive approximations are,

$$\begin{aligned} x_1 &= \varphi(x_0) = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.7+1}} = 0.76697 \\ x_2 &= \varphi(x_1) = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{0.76697+1}} = 0.75229 \\ x_3 &= \varphi(x_2) = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{0.75229+1}} = 0.75543 \\ x_4 &= \varphi(x_3) = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.75543+1}} = 0.75476 \\ x_5 &= \varphi(x_4) = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{0.75476+1}} = 0.75490 \\ x_6 &= \varphi(x_5) = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{0.75490+1}} = 0.75487 \\ x_7 &= \varphi(x_6) = \frac{1}{\sqrt{x_6+1}} = \frac{1}{\sqrt{0.75487+1}} = 0.75488 \\ x_8 &= \varphi(x_7) = \frac{1}{\sqrt{x_7+1}} = \frac{1}{\sqrt{0.75488+1}} = 0.75488 \end{aligned}$$

Since  $x_7 \approx x_8$  so the Iteration method gives no new values of  $x$  and the approximate root is correct to four decimal places. Hence the require root is 0.7548.

It's only for your clarification

Find a root of  $x^4 - x - 10 = 0$

[Graph]

Consider  $g1(x) = 10 / (x^3 - 1)$  and the fixed point iterative scheme  $x_{i+1} = 10 / (x_i^3 - 1)$ ,  $i = 0, 1, 2, \dots$  let the initial guess  $x_0$  be 2.0

|       |   |       |       |       |         |           |     |          |     |
|-------|---|-------|-------|-------|---------|-----------|-----|----------|-----|
| i     | 0 | 1     | 2     | 3     | 4       | 5         | 6   | 7        | 8   |
| $x_i$ | 2 | 1.429 | 5.214 | 0.071 | -10.004 | -9.978E-3 | -10 | -9.99E-3 | -10 |

So the iterative process with  $g1$  gone into an infinite loop without converging.

Consider another function  $g2(x) = (x + 10)^{1/4}$  and the fixed point iterative scheme  $x_{i+1} = (x_i + 10)^{1/4}$ ,  $i = 0, 1, 2, \dots$

let the initial guess  $x_0$  be 1.0, 2.0 and 4.0

|       |     |         |         |         |         |         |         |
|-------|-----|---------|---------|---------|---------|---------|---------|
| i     | 0   | 1       | 2       | 3       | 4       | 5       | 6       |
| $x_i$ | 1.0 | 1.82116 | 1.85424 | 1.85553 | 1.85558 | 1.85558 |         |
| $x_i$ | 2.0 | 1.861   | 1.8558  | 1.85559 | 1.85558 | 1.85558 |         |
| $x_i$ | 4.0 | 1.93434 | 1.85866 | 1.8557  | 1.85559 | 1.85558 | 1.85558 |

That is for  $g2$  the iterative process is converging to 1.85558 with any initial guess.

Consider  $g3(x) = (x+10)^{1/2} / x$  and the fixed point iterative scheme

$x_{i+1} = (x_i + 10)^{1/2} / x_i$ ,  $i = 0, 1, 2, \dots$

let the initial guess  $x_0$  be 1.8,

|       |     |        |         |         |         |         |         |     |        |
|-------|-----|--------|---------|---------|---------|---------|---------|-----|--------|
| i     | 0   | 1      | 2       | 3       | 4       | 5       | 6       | ... | 98     |
| $x_i$ | 1.8 | 1.9084 | 1.80825 | 1.90035 | 1.81529 | 1.89355 | 1.82129 | ... | 1.8555 |

That is for  $g3$  with any initial guess the iterative process is converging but very slowly to

Geometric interpretation of convergence with  $g1$ ,  $g2$  and  $g3$

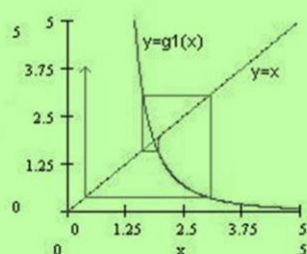


Fig g1

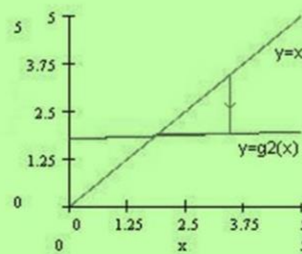


Fig g2

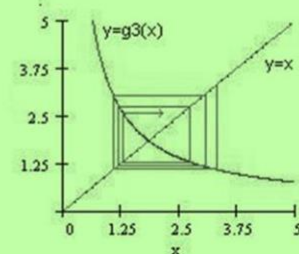


Fig g3

The graphs Figures Fig g1, Fig g2 and Fig g3 demonstrates the Fixed point Iterative Scheme with  $g1$ ,  $g2$  and  $g3$  respectively for some initial approximations. It's clear from the

- Fig g1, the iterative process does not converge for any initial approximation.
- Fig g2, the iterative process converges very quickly to the root which is the intersection point of  $y = x$  and  $y = g2(x)$  as shown in the figure.
- Fig g3, the iterative process converges but very slowly.

**Problem 02:**

Find the real root of the equation  $x - \ln x - 2 = 0$  that lies on  $[3, 4]$  using fixed point iteration method.

**Solution:**

Let  $f(x) = x - \ln x - 2 = 0$

Now  $f(3) = 3 - \ln 3 - 2 = -0.0986$

$f(4) = 4 - \ln 4 - 2 = 0.6137$

$f(3.5) = 3.5 - \ln 3.5 - 2 = 0.2472$

Hence there exist a root in  $(3, 3.5)$ .

Now we rewrite the given equation  $f(x) = 0$  in the following form:

$x = \ln x - 2 = \phi(x)$  [ say ]

$\phi'(x) = \frac{1}{x}$

Now  $\text{Max} (|\phi'(3)|, |\phi'(3.5)|) = (0.333, 0.2857) < 1$

Therefore  $\phi'(x) < 1$  in  $(3, 3.5)$ .

Then the iterative technique for fixed point iteration method is

$x_n = \phi(x_{n-1})$ , where  $n = 1, 2, 3, \dots, \text{etc.}$

Now let us start with the initial guess  $x_0 = 3$  then successive approximation using fixed point iteration method are tabulated below.

| Values of n | Values of $x_{n-1}$ | $x_n = \ln(x_{n-1}) + 2$ |
|-------------|---------------------|--------------------------|
| 1.          | 3                   | 3.098612289              |
| 2.          | 3.098612            | 3.130954362              |
| 3.          | 3.130954            | 3.141337866              |
| 4.          | 3.141338            | 3.144648781              |
| 5.          | 3.144649            | 3.145702209              |
| 6.          | 3.145702            | 3.146037143              |
| 7.          | 3.146037            | 3.146143611              |
| 8.          | 3.146144            | 3.146177452              |

Since  $|x_8 - x_7| \approx 0.000 = 0$ .

Hence the root of the given equation  $x - \ln x - 2 = 0$  is equal to 3.1461.

**Problem 02:**

**Problem 03:**

Find the real root of the equation  $x + \ln x - 2 = 0$  that lies on  $[1, 2]$  using fixed point iteration method.

**Solution:**

Let  $f(x) = x + \ln x - 2 = 0$

Now  $f(1) = 1 + \ln 1 - 2 = -1$

$f(2) = 2 + \ln 2 - 2 = 0.6931$

$f(1.5) = 1.5 + \ln 1.5 - 2 = -0.09453$

$f(1.7) = 1.7 + \ln 1.7 - 2 = 0.230628$

Hence there exist a root in  $(1.5, 1.7)$ .

Now we rewrite the given equation  $f(x) = 0$  in the following form:

$x = 2 - \ln x = \phi(x)$  [ say ]

$\phi'(x) = -\frac{1}{x}$

Now  $|\phi'(x)| = \left| -\frac{1}{x} \right| = \left| \frac{1}{x} \right| < 1$

Therefore  $\phi'(x) < 1$  in  $(1.5, 1.7)$ .

Then the iterative technique for fixed point iteration method is

$x_n = \phi(x_{n-1})$ , where  $n = 1, 2, 3, \dots, etc.$

Now let us start with the initial guess  $x_0 = 1.5$  then successive approximation using fixed point iteration method are tabulated below.

| Values of n | Values of $x_{n-1}$ | $x_n = 2 - \ln(x_{n-1})$ |
|-------------|---------------------|--------------------------|
| 1.          | 1.5                 | 1.594534892              |
| 2.          | 1.594535            | 1.53341791               |
| 3.          | 1.533418            | 1.572500828              |
| 4.          | 1.572501            | 1.547332764              |
| 5.          | 1.547333            | 1.563467349              |
| 6.          | 1.563467            | 1.553093986              |
| 7.          | 1.553094            | 1.559750939              |
| 8.          | 1.559751            | 1.555473846              |
| 9.          | 1.555474            | 1.558219777              |
| 10.         | 1.55822             | 1.556455999              |

|     |          |             |
|-----|----------|-------------|
| 11. | 1.556456 | 1.557588559 |
| 12. | 1.557589 | 1.556861171 |
| 13. | 1.556861 | 1.557328276 |
| 14. | 1.557328 | 1.557028291 |

Since  $|x_{14} - x_{13}| \approx 0.0001 = 0$ .

Hence the root of the given equation  $x - \ln x - 2 = 0$  is equal to 1.557328.

**Problem 04:**

Find the real root of the equation  $\sin x - 5x + 2 = 0$  that lies on  $[0, 1]$  using fixed point iteration method.

**Solution:**

Let  $f(x) = \sin x - 5x + 2 = 0$

$$f(x) = \sin\left(\frac{180x}{\pi}\right) - 5x + 2 = 0$$

Now  $f(0) = 2$

$$f(0.5) = \sin\left(\frac{0.5 \times 180}{\pi}\right) - 5 \times 0.5 + 2 =$$

$$f(1) = \sin\left(\frac{180}{\pi}\right) + 5.1 - 2 =$$

Hence there exist a root in  $(0, 0.5)$ .

Now we rewrite the given equation  $f(x) = 0$  in the following form:

$$x = \frac{\sin x + 2}{5} = \phi(x) \text{ [ say ]}$$

$$\phi'(x) = \frac{\cos x}{5}$$

$$\text{Now } |\phi'(x)| = \left| \frac{\cos x}{5} \right| < 1$$

Then the iterative technique for fixed point iteration method is

$$x_n = \phi(x_{n-1}), \text{ where } n = 1, 2, 3, \dots, \text{ etc.}$$

Now let us start with the initial guess  $x_0 = 0.5$  then successive approximation using fixed point iteration method are tabulated below.



| Values of n | Values of $x_{n-1}$ | $x_n = \frac{\left(\sin\left(\frac{180x_{n-1}}{\pi}\right)\right) + 2}{5}$ |
|-------------|---------------------|--|
| 1.          | 0.5                 | 0.327026799  |
| 2.          | 0.327026799         | 0.377578359  |
| 3.          | 0.377578359         | 0.341867524  |
| 4.          | 0.341867524         | 0.338731938  |
| 5.          | 0.338731938         | 0.338983858  |
| 6.          | 0.338983858         | 0.338223295  |

Since  $|x_6 - x_5| \approx 0.0002 = 0$ .

Hence the root of the given equation  $\sin x - 5x + 2 = 0$  is equal to 0.338983858.

## Try yourself:

### TYPE01:

To find the root of the following equations using Iteration method by taking your own guess:

- $x^3 - 5x + 1 = 0$
- $x^3 - 9x + 1 = 0$
- $x^3 + x - 5 = 0$
- $8x^3 - 2x - 1 = 0$
- $x^2 - 5x + 2 = 0$
- $2x - \log x = 7$
- $3x - \log_{10} x - 16 = 0$
- $e^x = 4x$
- $3x = 1 + \cos x$
- $x + \ln x - 2 = 0$
- $x^3 - 2x^2 - 4 = 0$
- $x^3 = 2x + 5$
- $e^x \tan x = 1$
- $4\sin x + x^2 = 0$
- $3x + \sin x = e^x$
- $\sin x = 1 - x^2$
- $\sin x = x^2 - 1$

### TYPE02:

- Find the root of  $x^2 + x - 1 = 0$  by iteration method given that root lies near 1.
- Find a real root  $\cos x = 3x - 1$  correct to three decimal places.
- Find by iteration method the root near 3.8 of equation  $2x - \log_{10} x = 7$  correct to four decimal places.

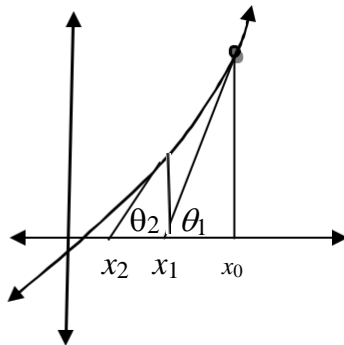
4. Solve the equation  $x^3 - 2x^2 - 5 = 0$  by iteration method.
5. Find the real root the equation  $x^3 + x^2 - 100 = 0$  by the method of successive approximations.
6. Find the root of  $x^2 = \sin x$  which lies between 0.5 and 1 correct to four decimal places.
7. Show that the equation  $\log_e x = x^2 - 1$  has exactly two real roots  $\alpha_1 = 0.45$  and  $\alpha_2 = 1$ .
8. Find the positive real root of  $x^3 + 3x^2 - 12x - 11 = 0$  correct to three decimal places.
9. Solve the equation  $x \sin x = 1$  for the positive root by iteration method.
10. Determine the real root of the equation  $\tan x = x$  by iteration method.
11. Find the root of  $x^2 + \ln x - 2 = 0$  in  $[1,2]$  by fixed point iteration method taking  $x_0 = 1$  correct to five decimal places.
12. Use the method of iteration to find a positive root between 0 and 1 of the equation  $xe^x = 1$ .
13. Compute a root of the equation  $e^x = x^2$  to an accuracy of  $10^{-5}$ , using the iteration method.

**TYPE03:**

1. Derive the fixed point iteration method to solve the equation  $f(x) = 0$ .
2. Write down the algorithm for fixed point iteration Method.
3. When does fixed point iteration Method Fails.

## Newton Raphson Method:

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Suppose we want to find a real root of the given equation  $f(x) = 0$  that lies in  $(a, b)$ . Consider  $x_0 \in (a, b)$  be an arbitrary point which is very close to the desired root of the given equation  $f(x) = 0$ . Draw a tangent to the curve  $f(x) = 0$  at  $x = x_0$ . Suppose this tangent makes an angle  $\theta_1$  with x-axis at the point  $(x_1, 0)$  where  $x_1$  is the first approximation of the desired root.

$$\tan \theta_1 = \frac{f(x_0)}{x_0 - x_1} \dots\dots\dots(i)$$

On the other hand, the slope of the curve  $f(x) = 0$  at  $x = x_0$  is  $f'(x_0)$ .

$$\tan \theta_1 = f'(x_0) \dots\dots\dots(ii)$$

Therefore, from equation (i) and (ii) we have

$$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If  $f(x_1) = 0$  we say that  $x_1$  is the desired root of the given equation  $f(x) = 0$ .

Suppose that  $f(x_1) \neq 0$ . Now draw a tangent to the curve  $f(x) = 0$  at  $x = x_1$  which makes an angle  $\theta_2$  with the x-axis at the point  $(x_2, 0)$  where  $x_2$  is the second approximation of the desired root. Consequently we have

$$\tan \theta_2 = f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

On simplification we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general the k-th approximation  $x_k$  can be computed by using the following iterative

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}, \quad k = 3, 4, 5, \dots \text{etc.}$$

We shall continue this iterative process until the value of two successive approximations are approximately equal, i.e.  $x_k \approx x_{k-1}$  or  $f(x_k) \approx 0$ .

**Advantage of Newton Raphson method:**

1. Converge fast if it converge to the root compare to another method.
2. Requires only one guess.
3. Convergence to the root quadratically.
4. Easy to convert to multiple dimensions.
5. Can be to polish a root found by another methods.

**Dis-advantage / drawback of Newton Raphson method:**

1. Must find the derivative.
2. Poor global convergence properties.
3. It takes more computing time
4. It should never be used when the graph of  $f(x) = 0$  is nearly horizontal where it crosses the x-axis.
5. Dependent on initial guess
  - May be too far from local root
  - May encounter a zero derivative
  - May loop indefinitely

**Algorithm for Newton Rapson method:**

| Steps | Task  |
|-------|---|
| 1     | Define $f(x)$   |
| 02    | Define $f'(x)$  |
| 03    | Read $x_0$  |
| 4     | Set $k = 0$   |
| 5     | $k = k + 1$   |
| 06    | Calculate $x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$                    |
| 7     | If $ x_k - x_{k-1}  \leq 0.0001$ then<br>GOTO Step 8<br>elseif<br>GOTO Step 5 |
| 8     | Print $x_k$ , the desired root  |
| 9     | STOP  |

**Problem 01:**

Find the root of the equation  $x^3 - 3x - 5 = 0$  by Newton-Rapshon Method correct to four decimal places.

Solution:

Let  $f(x) = x^3 - 3x - 5$  then  $f'(x) = 3x^2 - 3$ .

Here  $f(2) = 8 - 6 - 5 = -3 < 0$  and  $f(3) = 27 - 9 - 5 = 13 > 0$

Since  $f(2)$  and  $f(3)$  are of opposite sign so at least one real root lies between 2 and 3.

we know that from Newton-Rapshon method ,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x^3 - 3x - 5}{3x^2 - 3} \quad [\text{putting values}]$$

$$x_{n+1} = \frac{3x^3 - 3x - x^3 + 3x + 5}{3x^2 - 3}$$

$$x_{n+1} = \frac{2x^3 + 5}{3x^2 - 3} \dots\dots\dots (1)$$

Choosing an initial guess  $x_0 = 2$  and putting  $n = 0$  and  $x_0 = 2$  in above mentioned equation (1), we are capable to find the successive improved approximations are as follows:

$$x_1 = \frac{2x_0^3 + 5}{3x_0^2 - 3} = \frac{2 \times 2^3 + 5}{3 \times 2^2 - 3} = 2.333$$

$$x_2 = \frac{2x_1^3 + 5}{3x_1^2 - 3} = \frac{2 \times (2.333)^3 + 5}{3 \times (2.333)^2 - 3} = 2.2806$$

$$x_3 = \frac{2x_2^3 + 5}{3x_2^2 - 3} = \frac{2 \times (2.2806)^3 + 5}{3 \times (2.2806)^2 - 3} = 2.2790$$

$$x_4 = \frac{2x_3^3 + 5}{3x_3^2 - 3} = \frac{2 \times (2.2790)^3 + 5}{3 \times (2.2790)^2 - 3} = 2.2790$$

Since  $x_4 = x_3$  so the Newton Rapshon method gives no new values of x and the approximate root is correct to four decimal places.Hence the require root is 2.2790 .

**Problem 02:**

Using Newton-Rapshon method ,find the root of the equation  $x^4 - x - 10 = 0$  which is nearer to  $x = 2$  correct to three decimal places.

Solution:

Let  $f(x) = x^4 - x - 10$  then  $f'(x) = 4x^3 - 1$  .

Here  $f(1) = 1 - 1 - 10 = -10 < 0$  and  $f(2) = 16 - 2 - 10 = 4 > 0$

Since  $f(1)$  and  $f(2)$  are of opposite sign so at least one real root lies between 1 and 2.

we know that from Newton-Rapshon method ,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1}$$

$$x_{n+1} = \frac{4x_n^4 - x_n + x_n^4 + x_n + 10}{4x_n^3 - 1}$$

$$x_{n+1} = \frac{3x_n^4 + 10}{4x_n^3 - 1} \dots\dots\dots(1)$$

Choosing an initial guess  $x_0 = 1.9$  and putting  $n = 0$  and  $x_0 = 1.9$  in above mentioned equation (1), we are capable to find the sccessive improved approrimations are as follows:

$$x_1 = \frac{3x_0^4 + 10}{4x_0^3 - 1} = \frac{3 \times (1.9)^4 + 10}{4 \times (1.9)^3 - 1} = 1.8$$

$$x_2 = \frac{3x_1^4 + 10}{4x_1^3 - 1} = \frac{3 \times (1.8)^4 + 10}{4 \times (1.8)^3 - 1} = 1.85556$$

$$x_3 = \frac{3x_2^4 + 10}{4x_2^3 - 1} = \frac{3 \times (1.85556)^4 + 10}{4 \times (1.85556)^3 - 1} = 1.85556$$

Since  $x_2 = x_3$  so the Newton Rapshon method gives no new values of x and the approximate root is correct to five decimal places.Hence the require root is 1.85556 .

**Problem 03:**

Find the real root of the equation  $x^2 - 4\sin x = 0$  correct to four decimal places using Newton-Rapshon method.

Solution:

Let  $f(x) = x^2 + 4\sin x$  then  $f'(x) = 2x + 4\cos x$ .

Here  $f(-1) = (-1)^2 + 4\sin(-1) = -2.36 < 0$  and  $f(-2) = (-2)^2 + 4\sin(-2) = 0.36 < 0$ .

Hints: Calculator must be in radian Mode.

Since  $f(-1)$  and  $f(-2)$  are of opposite sign so at least one real root lies between -2 and -1. we know that from Newton-Rapshon method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 + 4\sin x_n}{2x_n + 4\cos x_n}$$

$$x_{n+1} = \frac{2x_n^2 + 4x_n\cos x_n - x_n^2 - 4\sin x_n}{2x_n + 4\cos x_n}$$

$$x_{n+1} = \frac{x_n^2 + 4x_n\cos x_n - 4\sin x_n}{2x_n + 4\cos x_n} \dots\dots\dots(1)$$

Choosing an initial guess  $x_0 = -1.9$  and putting  $n = 0$  and  $x_0 = -1.9$  in above mentioned equation (1), we are capable to find the successive improved approximations are as follows:

$$x_1 = \frac{x_0^2 + 4x_0\cos x_0 - 4\sin x_0}{2x_0 + 4\cos x_0}$$

$$= \frac{(-1.9)^2 + 4 \times (-1.9) \cos(-1.9) - 4 \sin(-1.9)}{2 \times (-1.9) + 4 \cos(-1.9)} = -1.93$$

$$x_2 = \frac{x_1^2 + 4x_1\cos x_1 - 4\sin x_1}{2x_1 + 4\cos x_1}$$

$$= \frac{(-1.93)^2 + 4 \times (-1.93) \cos(-1.93) - 4 \sin(-1.93)}{2 \times (-1.93) + 4 \cos(-1.93)} = -1.9338$$

Since  $x_1 \approx x_2$  so the Newton Rapshon method gives no new values of x and the approximate root is correct to two decimal places. Hence the require root is -1.93 .

**Problem 04:**

Find the root of the equation  $x \sin x + \cos x = 0$  ,using Newton-Rapshom method.

Solution:

Let  $f(x) = x \sin x + \cos x$  then  $f'(x) = 1.\sin x + x \cos x + (-\sin x) = x \cos x$ .

Here  $f(2) = 2\sin 2 + \cos 2 = 1.40 > 0$  and  $f(3) = 3\sin 3 + \cos 3 = -0.56 > 0$ .

Hints: Calculator must be in radian Mode.

Since  $f(2)$  and  $f(3)$  are of opposite sign so at least one real root lies between 2 and 3.

we know that from Newton-Rapshon method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n^2 \cos x_n - x_n \sin x_n - \cos x_n} \dots \dots \dots (1)$$

Choosing an initial guess  $x_0 = 2.79$  and putting  $n = 0$  and  $x_0 = 2.79$  in above mentioned equation (1), we are capable to find the successive improved approximations are as follows:

$$\begin{aligned} x_1 &= \frac{x_0^2 \cos x_0 - x_0 \sin x_0 - \cos x_0}{x_0 \cos x_0} \\ &= \frac{(2.79)^2 \cos(2.79) - 2.79 \sin(2.79) - \cos(2.79)}{2.79 \cos(2.79)} \\ &= 2.7984 \\ x_2 &= \frac{x_1^2 \cos x_1 - x_1 \sin x_1 - \cos x_1}{x_1 \cos x_1} \\ &= \frac{(2.7984)^2 \cos(2.7984) - 2.7984 \sin(2.7984) - \cos(2.7984)}{2.7984 \cos(2.7984)} \\ &= 2.79834 \end{aligned}$$

Since  $x_1 \approx x_2$  so the Newton Rapshon method gives no new values of  $x$  and the approximate root is correct to three decimal places. Hence the require root is 2.7984 .

**Try yourself:**

**TYPE01:**

To find the root of the following equations using Newton-Rapshon method by taking your own guess:

1.  $x + \log x = 2$
2.  $2x = \log_{10} x + 7$
3.  $e^x = 4x$
4.  $x^3 + x^2 - 1 = 0$
5.  $3x - \cos x - 1 = 0$
6.  $\sin x = 1 - x^2$
7.  $\sin^2 x = x^2 - 1$
8.  $3x + \sin x = e^x$
9.  $\sin x = 10(x - 1)$
10.  $e^x \tan x = 1$
11.  $x \sin x = 1$
12.  $\cos x - xe^x = 0$
13.  $x^3 - 2x^2 - 4 = 0$

**TYPE02:**

1. Find the real root of  $2x - \log_{10} x - 7 = 0$  using Newton Raphson in (3, 4).
2. Using Newton Raphson Method find root of the equation  $e^x - 4x^2 = 0$  that lies in (4, 5).
3. Using Newton Raphson Method find root of the equation  $e^{x^2} - 4\sin x = 0$  in the interval (0, 1).
4. Find a real root of the equation  $x^3 + x^2 - 1 = 0$  by using Newton Raphson Method correct up to four decimal places.
5. Use Newton Raphson's Method to find the root of  $x^3 - x - 2 = 0$  with  $x_0 = 3$ .
6. Find a real root of the equation  $x^4 + x^2 - 80 = 0$  by using Newton Raphson Method correct up to three decimal places.
7. Find a real root of the equation  $x^2 + \ln x - 2 = 0$  in [1,2] by using Newton Raphson Method correct up to five decimal places.
8. By using Newton Method find a real root of the equation  $x^4 - x - 10 = 0$  which is near to  $x=2$  correct up to three decimal places.
9. Find a real root of the equation  $x^3 + x^2 + 3x + 4 = 0$  by using Newton Raphson Method correct up to four decimal places.
10. Find by using Newton Method the real root of the equation  $e^x = 4x$  which is approximately 2 correct to three places of decimals.

**TYPE03:**

1. Using Newton Raphson Method establish the formula  $x_{n+1} = \frac{1}{2} \left( x_n + \frac{N}{x_n} \right)$  to calculate the square root of N. Hence find the square root of 5 correct to four places of decimals.
2. Show that the iterative formula for finding the reciprocal of N is  $x_{n+1} = x_n (2 - Nx_n)$  and hence find the value of  $31^{-1}$ .
3. Derive the iterative formula for Newton Raphson method to solve the equation  $f(x) = 0$ .
4. Write down the merits and demerits of Newton Raphson Method.
5. Write down the algorithm for Newton Raphson Method.
6. When does Newton Raphson Method Fails.