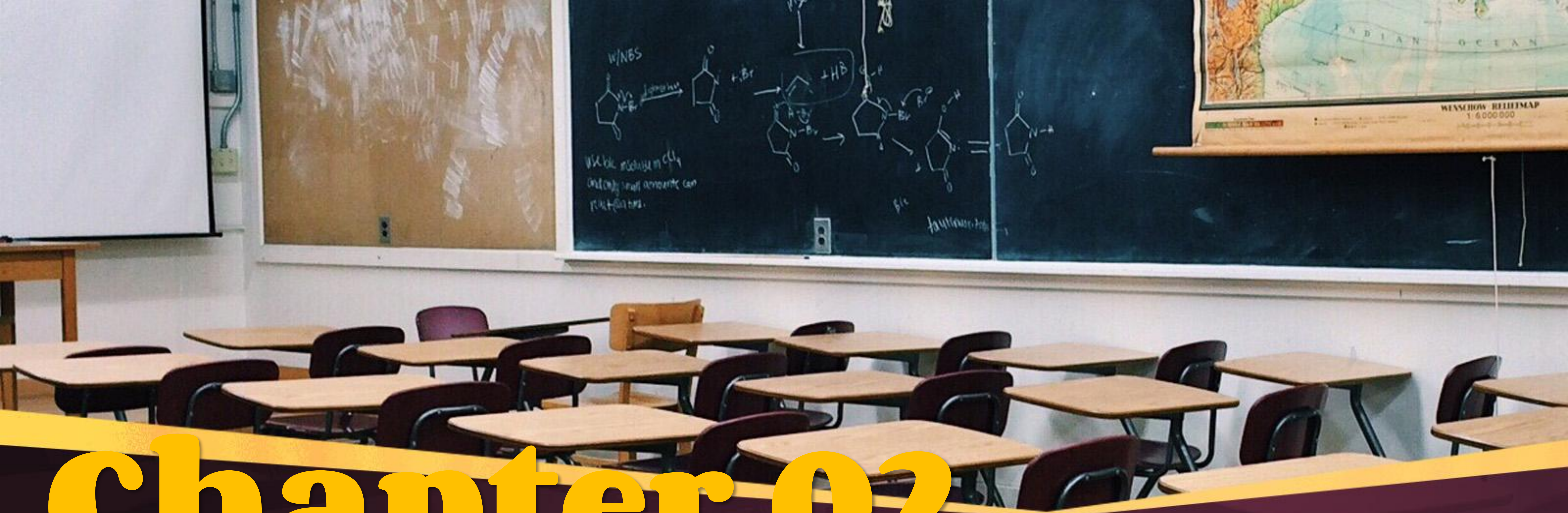


Chapter 02

Solution of Algebraic and Transcendental Equations

2.Fixed Point Iteration Method

Procedure of Fixed Point Iteration Method:

Step 1 : The given function = f (x)

$$Let, x^3 + x^2 - 1 = 0$$

$$Given, f(x) = x^3 + x^2 - 1$$

Find f (x) in the step 1

Procedure of Fixed Point Iteration Method:

$For, b = x = 2 \therefore f(b) = b^3 + b^2 - 1$

$\therefore f(2) = 2^3 + 2^2 - 2 = +10 > 0$

$For, a = x = -1, f(a) = a^3 + a^2 - 1$

$\therefore f(-1) = (-1)^3 + (-1)^2 - 1 = -1 < 0$

Step 1 : The given function = f (x)

$Let, x^3 + x^2 - 1 = 0$

$Given, f(x) = x^3 + x^2 - 1$

Step 2 : Choose , two real numbers a and b
Such that , f (a) * f (b) < 0

$For, a = 0, f(a) = a^3 + a^2 - 1$

$\therefore f(0) = 0^3 + 0^2 - 1 = -1$

$For, b = 1 \therefore f(b) = b^3 + b^2 - 1$

$\therefore f(1) = 1^3 + 1^2 - 1 = +1 > 0$

$$\therefore f(a) \times f(b) = f(0) \times f(1) = (-1) \times 1 = -1 < 0$$

Find the Interval between (a , b) by the step 2

Since , f (a) = f(0) is negative and f (b) = f (1) is positive
So at least one real root lies between 0 and 1.

Step 3 :

Convert from the given equation f (x) = 0 into the form of x = g(x) .

$$\therefore x = g(x) ....(1)$$

The given equation is

$$x^3 + x^2 - 1 = 0$$

$$\Rightarrow x^2 (x+1) = 1$$

$$\Rightarrow x^2 = \frac{1}{(x+1)}$$

$$\Rightarrow x = \frac{1}{\sqrt{(x+1)}}$$

Find out x from the given equation and then give another name like g(x) or $\Phi(x)$...

$$\therefore g(x) = x = \frac{1}{\sqrt{(x+1)}} (say)$$

$$\therefore x = g(x) = \frac{1}{\sqrt{1+x}} ....(1)$$

Step 4 : *Find* $g'(x)$ *and* $|g'(x)|<1$ *for* $x\in(a,b)$

Find $g'(x)$ *and check* $|g'(x)|<1$ *for* $x\in(a,b)$

$$\because g(x)=\frac{1}{\sqrt{(x+1)}}$$

$$\therefore \frac{d}{dx}g(x)=\frac{d}{dx}\frac{1}{\sqrt{(x+1)}}$$

$$\therefore g'(x)=\frac{d}{dx}(x+1)^{-\frac{1}{2}}=-\frac{1}{2}(x+1)^{-\frac{1}{2}-1}$$

$$=-\frac{1}{2}(x+1)^{-\frac{3}{2}}$$

$$g'(x)=-\frac{1}{2}(1+x)^{-\frac{3}{2}}$$

$$\therefore |g'(x)|=\left|-\frac{1}{2}(1+x)^{-\frac{3}{2}}\right|$$

we can take, $x=0.5$ *then*,

$$\therefore |g'(0.5)|=\left|-\frac{1}{2}(1+0.5)^{\frac{3}{2}}\right|<1 \text{ for } x\in(0,1)$$

Therefore, the iteration method is applicable for the given function.

Step 5: *Let the initial value,* $x = x_0$ *for the given function* $f(x) = 0$

we substitute the value of $x = x_0$ in the right-hand side of the equation (1)

Then we get , First approximate value, $x_1 = g(x_0)$

$$\therefore\ x = g(x) = \frac{1}{\sqrt{1+x}} \text{.....(1)}$$

Find the first approximate value from the R.H.S of equ. (1) by using (a , b)

Let, initial value, $x = x_0 = 0$, *here* $0 \in (0,1)$

then the R.H.S of equation (1) becomes,

First approximate value, $x_1 = g(x_0) = \frac{1}{\sqrt{1+x_0}}$

$$for\ x_0 = 0\ \therefore\ x_1 = g(0) = \frac{1}{\sqrt{1+0}} = 1$$

Step 5 (Remain Part) :

Again, substituting the value of $x = x_1$ in the equation of (1)

Then we get , 2nd approximate value, $x_2 = g(x_1)$

Similarly find the 2nd approximate value

$$First\ approximate\ value,\ x_1 = g(0) = \frac{1}{\sqrt{1+0}} = 1$$

$$for\ \ x = x_1 = 1\ \ \therefore x_2 = g(x_1) = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+1}} = 0.7071$$

$$\therefore 2nd\ \ approximate\ value, x_2 = 0.7071$$

Step 5 (Remain Part) :

Similarly ,

$$x_3 = g(x_2)$$

$$x_4 = g(x_3)$$

$$x_3 = g(x_2) = \frac{1}{\sqrt{1 + x_2}}$$

$$for\ \ x = x_2 = 0.7071$$

$$\therefore x_3 = g(x_2) = \frac{1}{\sqrt{1 + 0.7071}} = \frac{1}{\sqrt{1.7071}} = 0.7654$$

$$\therefore x_4 = g(x_3) = \frac{1}{\sqrt{1 + 0.7654}} = \frac{1}{\sqrt{1.7654}} = 0.7526$$

Similarly find the 3rd & 4-th approximate value

Step 5 (Remain Part) :

Similarly , $x_5 = g(x_4)$

....

....

....

$x_n = g(x_{n-1})$

Therefore the iterative formula for successive approximation method is,

$$\therefore x_n = g(x_{n-1}) \text{ for } n = 1, 2, 3, 4, ...etc$$

Here, x_n *is the* $n-th$ *approximation of the desired root of* $f(x) = 0$.

$$x_5 = g(x_4) = \frac{1}{\sqrt{1 + x_4}}$$

for $x = x_4 = 0.7526$

$$\therefore x_5 = g(x_4) = \frac{1}{\sqrt{1 + 0.7526}} = \frac{1}{\sqrt{1.7526}} = 0.7554$$

$$\therefore x_6 = g(x_5) = \frac{1}{\sqrt{1 + 0.7554}} = \frac{1}{\sqrt{1.7554}} = 0.7548$$

$$\therefore x_7 = g(x_6) = \frac{1}{\sqrt{1 + 0.7548}} = \frac{1}{\sqrt{1.7548}} = 0.7549$$

$$\therefore x_8 = g(x_7) = \frac{1}{\sqrt{1 + 0.7549}} = \frac{1}{\sqrt{1.7548}} = 0.7549$$

Step 6 : We shall continue this iterative cycle until the values of two successive approximations are almost equal.

$$\therefore x_7 = g(x_6) = \frac{1}{\sqrt{1+0.7548}} = \frac{1}{\sqrt{1.7548}} = 0.7549$$

$$\therefore x_8 = g(x_7) = \frac{1}{\sqrt{1+0.7549}} = \frac{1}{\sqrt{1.7548}} = 0.7549$$

$$\sin ce, x_7 \approx x_8$$

Hence the require root is 0.7549.

This above mentioned method is known as **Iteration method. Or Method of successive approximation or Fixed point Iteration.**

Problem 1 : Find the real root of the equation $\text{Sin}\,x-5x+2=0$ that lies on [0, 1] using fixed point iteration method.

Solution: Given That , $\text{Sin}\,x-5x+2=0$

You have to do mode on radian to your calculator.

$$Let,\ f(x)=\text{Sin}\,x-5x+2=0$$

$$\therefore\ f(x)=\text{Sin}\,x-5x+2$$

$$For\ x=0,\ f(0)=\sin 0-5\times 0+2=2>0$$

$$For\ x=1,\ f(1)=\sin 1-5\times 1+2=-2.1585290152<0$$

Since f (0) and f (1) are the opposite sign , So the root lies on [0, 1]

Now we rewrite the equation f(x) = 0

Step 2:

$$\text{Sin}\,x - 5x + 2 = 0$$

$$\Rightarrow \text{Sin}\,x - 5x + 2 = 0$$

$$\Rightarrow -5x = -2 - \text{Sin}\,x$$

$$\Rightarrow 5x = 2 + \text{Sin}\,x$$

$$\therefore\ x = \frac{2 + \text{Sin}\,x}{5}$$

$$\because\ x = \frac{2 + \text{Sin}\,x}{5}$$

$$\therefore\ g(x) = x = \frac{2 + \text{Sin}\,x}{5} \ldots\ldots\ldots.(1)$$

Step 3 : Test

$$\therefore g(x) = \frac{2 + \text{Sin}\,x}{5}$$

$$\frac{d}{dx}g(x) = \frac{d}{dx}\left(\frac{2 + \text{Sin}\,x}{5}\right) \Rightarrow g'(x) = \frac{1}{5}\frac{d}{dx}(2 + \text{Sin}\,x)$$

$$= \frac{1}{5}\left(\frac{d}{dx}2 + \frac{d}{dx}\text{Sin}\,x\right)$$

$$= \frac{1}{5}(0 + \cos x) = \frac{1}{5}\cos x$$

$$\therefore\ g'(x) = \frac{1}{5}\cos x$$

$$\therefore\ |g'(x)| = \left|\frac{1}{5}\cos x\right|$$

$$= \left|\frac{1}{5}\cos 0\right| \ \textit{for}\ (0,1)$$

$$\therefore\ |g'(x)| = \left|\frac{1}{5}.1\right| < 1 \ \textit{for}\ (0,1)$$

You have to do mode on radian to your calculator

Step 4:

We take Initial Value, $x = x_0 = 0.5$, then successive approximation using fixed point iteration method are tabulated below.

Step 5:

Values of n	$Values of\ x_n$	$x_n = g(x_n) = \frac{2 + \text{Sin}\, x_n}{5}$
01	0.5	0.495885107
02	0.495885107	0.4951620663
03	0.4951620663	0.4950348517
04	0.4950348517	0.495012463
05	0.495012463	0.4950085238
06	0.4950085238	0.4950078304

$$\text{Sin}\,ce\,|x_6 - x_5| \approx 0.000001 = 0$$

Hence the root of the given equation is equal to **0.4950078304**

.

Problem 2:

Find the real root of the equation x- lnx - 2 = 0 that lies on [3, 4] using fixed point iteration method.

Solution: Let , $f(x) = x - \ln x - 2 = 0$

$$For\ x = 3, f(3) = 3 - \ln 3 - 2$$
$$= -0.0986 < 0$$

$$For\ x = 4, f(4) = 4 - \ln 4 - 2$$
$$= 0.61370 > 0$$

Hence there exist a root in (3, 4).

Now,

$$x - \ln x - 2 = 0$$
$$\Rightarrow x = \ln x + 2$$
$$\therefore g(x) = x = \ln x + 2\ (say)$$

$$\therefore g'(x) = \frac{1}{x} + 0$$

$$\therefore g'(x) = \frac{1}{x}$$

$For\,(3,4)$

$$\therefore \left|g'(3)\right| = \left|\frac{1}{3}\right| < 1$$

Now let the initial value , $x = x_0 = 3$

Then successive approximation using fixed point iteration method are tabulated below.

Values of n	$Values of\ x_n$	$x_n = g(x_n) = \ln x_n + 2$
01	03	3.098612289
02	3.098612	3.130954362
03	3.130954	3.141337866
04	3.141337	3.144648781
05	3.144648	3.145702209
06	3.145702	3.146037143
07	3.146037	3.146143611
08	3.146143	3.146177452

$$Since\ |x_8 - x_7| \approx 0.0000 = 0$$

Hence the root of the given equation is equal to 3.1461

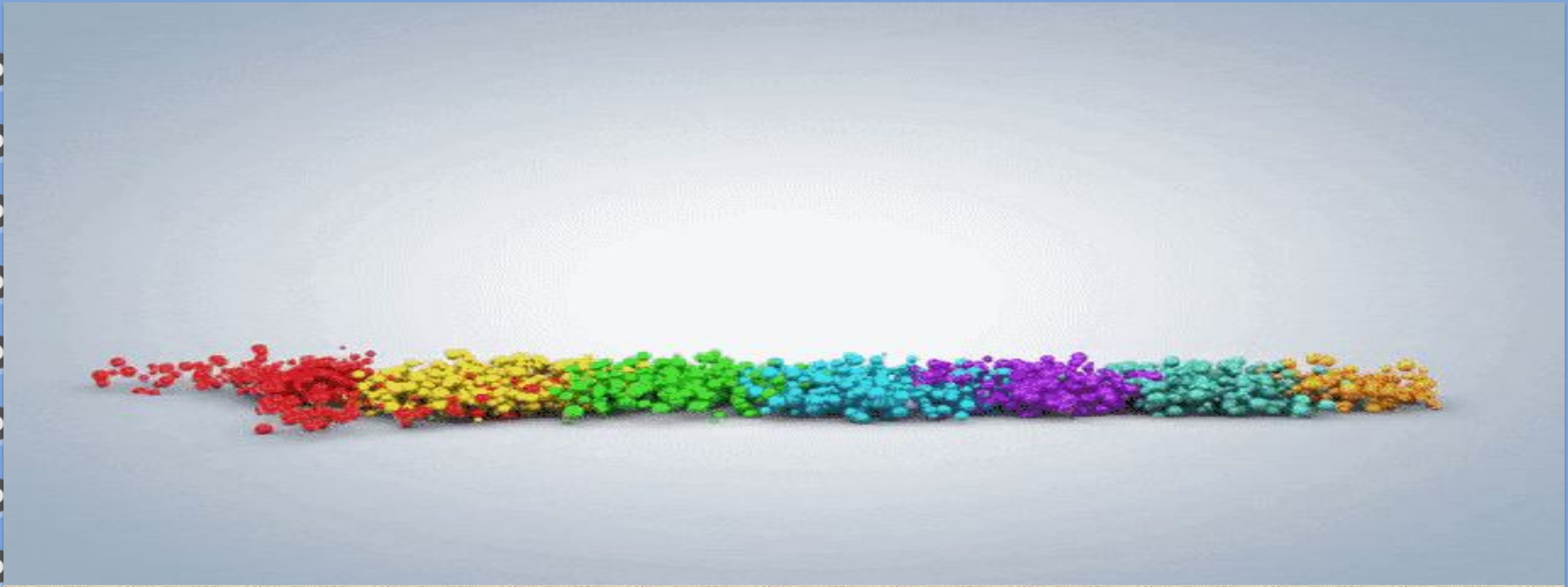
Algorithm for Iteration method:

Steps	Task
01	Define g(x)
02	Read initial value, x_0
03	Set n =1
04	$x_n = g(x_{n-1})$
05	If $\lvert x_n - x_{n-1} \rvert \approx 0.0001$ $Then\ go\ to\ step\ 6$ $else$ $n = n + 1$ $Go\ to\ step\ 4.$
06	Print x_n ,the desired root
07	Stop

Practice Work

Find the root of the following equation by using Iteration method :

1. $2x = \ln x + 7$ *correct to 4 decimal places*
2. $3x + \sin x = e^x$ *correct to 3 decimal places.*
3. $\cos x = 3x - 1$ *correct to 3 decimal places.*
4. $e^x \tan x = 1$ correct to 3 decimal
5. $3x = \ln_{10} x + 7$ *correct to 4 decimal places*



For Stay With Me