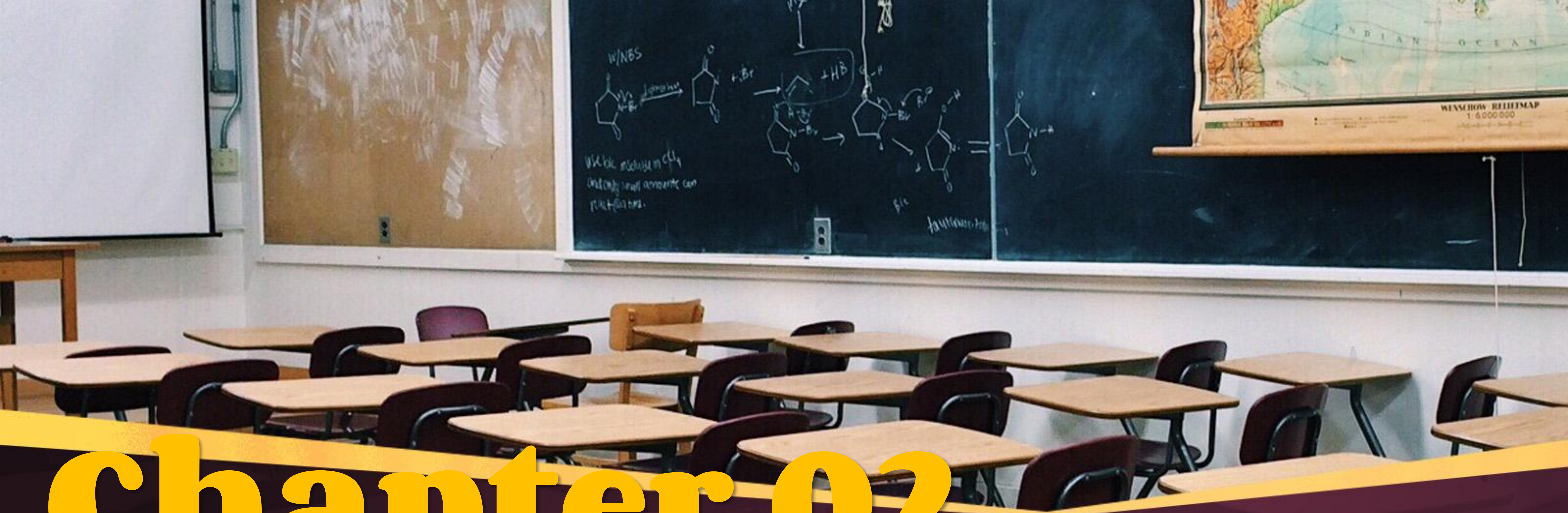


# Chapter 02

## Solution of Algebraic and Transcendental Equations

# 3. Newton - Raphson Method

# Procedure of Newton – Raphson Method:

**Step 1 :** The given function = f (x) and find $f'(x)$

$$Given, x^3 - 3x - 5 = 0$$

$$let, f(x) = x^3 - 3x - 5$$

**Find f (x) and $f'(x)$ in the step 1**

$$f(x) = x^3 - 3x - 5$$

$$\therefore \frac{d}{dx} f(x) = \frac{d}{dx} x^3 - 3\frac{d}{dx} x - \frac{d}{dx} 5$$

$$\therefore f'(x) = 3x^2 - 3 \times 1 - 0$$

$$= 3x^2 - 3$$

# Procedure of Newton – Raphson Method:

Find the Interval between ( a , b ) in the step 2

**Step 2 :** Choose , two real numbers a and b $such\ that, f(a) \times f(b) < 0$

$$f(x) = x^3 - 3x - 5$$

$$\therefore\ for\ x = a = 2,\ f(a) = a^3 - 3 \times a - 5$$

$$\therefore f(2) = 2^3 - 3 \times 2 - 5$$

$$= 8 - 11 = -3 < 0$$

$$f(x) = x^3 - 3x - 5$$

$$\therefore\ for\ x = b = 3, f(b) = b^3 - 3 \times b - 5$$

$$\therefore f(b) = b^3 - 3 \times b - 5$$

$$= 27 - 9 - 5 = 13 > 0$$

$$\therefore f(a) \times f(b) = f(2) \times f(3) = (-3) \times 13 = -39 < 0$$

Since , f (a) = f ( 2) is negative and f ( b ) = f (3) is positive, So at least one real root lies between 2 and 3.

**Step 3 :** Find the follwing equation (1) by Newton – Raphson formula.

**Newton – Raphson formula**

we know that from Newton-Rapshon formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{........}(1)$$

$$\because f(x) = x^3 - 3x - 5$$

$$\therefore f(x_n) = x^3_n - 3x_n - 5$$

$$\because f'(x) = 3x^2 - 3$$

$$\therefore f'(x_n) = 3x^2_n - 3$$

$$\because x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_{n+1} = x_n - \frac{x^3_n - 3x_n - 5}{3x^2_n - 3} \text{..........}(1)$$

**Step 4 :** *Let, the initial value*, $x = x_0 \in (a, b)$ *and putting the value*, $n = 0$ in *the above equation* (1)

we are capable to find the successive improved approximations are as follows:

$$\because x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{........}(1)$$

*putting* $n = 0$, *and*

*let the initial value*, $x = x_0$ *in equation* (1)

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Putting , n= 0 and then take initial value and then find first approximate value.

$$x_{n+1} = x_n - \frac{x^3{}_n - 3x_n - 5}{3x^2{}_n - 3} \text{......}(1)$$

*For*, $n = 0$ *and* $x = x_0 = 2$ *(say)* *in equation* (1)

$$x_{0+1} = x_0 - \frac{x^3{}_0 - 3x_0 - 5}{3x^2{}_0 - 3}$$

$$\therefore x_1 = x_0 - \frac{x^3{}_0 - 3x_0 - 5}{3x^2{}_0 - 3}$$

$$\because x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{........}(1)$$

$putting\ n = 0, let\ the\ initial\ value, x = x_0\ in\ equation (1)$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Putting , n= 0 and then take initial value and then find first approximate value.

### Step 4: Remain part

$$\therefore x_1 = x_0 - \frac{x^3_0 - 3x_0 - 5}{3x^2_0 - 3}$$

$For\ x = x_0 = 2 (say); [From\ interval (2,3)]$

$$x_1 = x_0 - \frac{x^3_0 - 3x_0 - 5}{3x^2_0 - 3}$$

$$= 2 - \frac{2^3 - 3\times 2 - 5}{3\times 2^2 - 3} = 2.333$$

## Step 4: Remain part

$$\because\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{........(1)}$$

*putting* $n = 1$, *let the initial value*, $x = x_1$ *in equation* (1)

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\therefore\ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Similarly find the others approximate value.

$$x_{n+1} = x_n - \frac{x^3_n - 3x_n - 5}{3x^2_n - 3} \quad \text{.......(1)}$$

*Again, putting* $n = 1$, $x = x_1$ *in equation* (1)

$$x_2 = x_1 - \frac{x^3_1 - 3x_1 - 5}{3x^2_1 - 3}$$

*For* $x = x_1 = 2.333$

$$x_2 = x_1 - \frac{x^3_1 - 3x_1 - 5}{3x^2_1 - 3}$$

$$= 2.333 - \frac{(2.333)^3 - 3\times 2.333 - 5}{3\times(2.333)^2 - 3} = 2.2806$$

Step 4: Remain part

$$Let\ see\ the\ above\ data\ by\ the\ following\ table:$$

| No. of Iterations, n | $x_{n+1}=x_n-\frac{x^3_n-3x_n-5}{3x^2_n-3}$ | $x_{n+1}=x_n-\frac{x^3_n-3x_n-5}{3x^2_n-3}$ |
|---|---|---|
| $n=0$ | $x_1=x_0-\frac{x^3_0-3x_0-5}{3x^2_0-3}$ | $x_1=2-\frac{2^3-3\times2-5}{3\times2^2-3}=2.333$ |
| $n=1$ | $x_2=x_1-\frac{x^3_1-3x_1-5}{3x^2_1-3}$ | $x_2=2.333-\frac{(2.333)^3-3\times2.333-5}{3\times(2.333)^2-3}=2.2806$ |

## Step 4: Remain part

*Similarly,*

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

... ... ....

... ... ....

... ... ....

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

| No. of Iterations, n | $x_{n+1} = x_n - \frac{x^3{}_n - 3x_n - 5}{3x^2{}_n - 3}$ | $x_{n+1} = x_n - \frac{x^3{}_n - 3x_n - 5}{3x^2{}_n - 3}$ |
|---|---|---|
| $n = 2$ | $x_3 = x_2 - \frac{x^3{}_2 - 3x_2 - 5}{3x^2{}_2 - 3}$ | $x_3 = 2.2806 - \frac{(2.2806)^3 - 3\times(2.2806) - 5}{3\times(2.2806)^2 - 3} = 2.2790$ |
| $n = 3$ | $x_4 = x_3 - \frac{x^3{}_3 - 3x_3 - 5}{3x^2{}_3 - 3}$ | $x_4 = 2.2790 - \frac{(2.2790)^3 - 3\times 2.2790 - 5}{3\times(2.2790)^2 - 3} = 2.2790$ |

**Step 5 :** We shall continue this iterative process until the value of two successive approximation are approximately equal.

$$That\ is\ ,\ x_n \approx x_{n-1}\ \ or\ f(x_n) \approx 0\ .$$

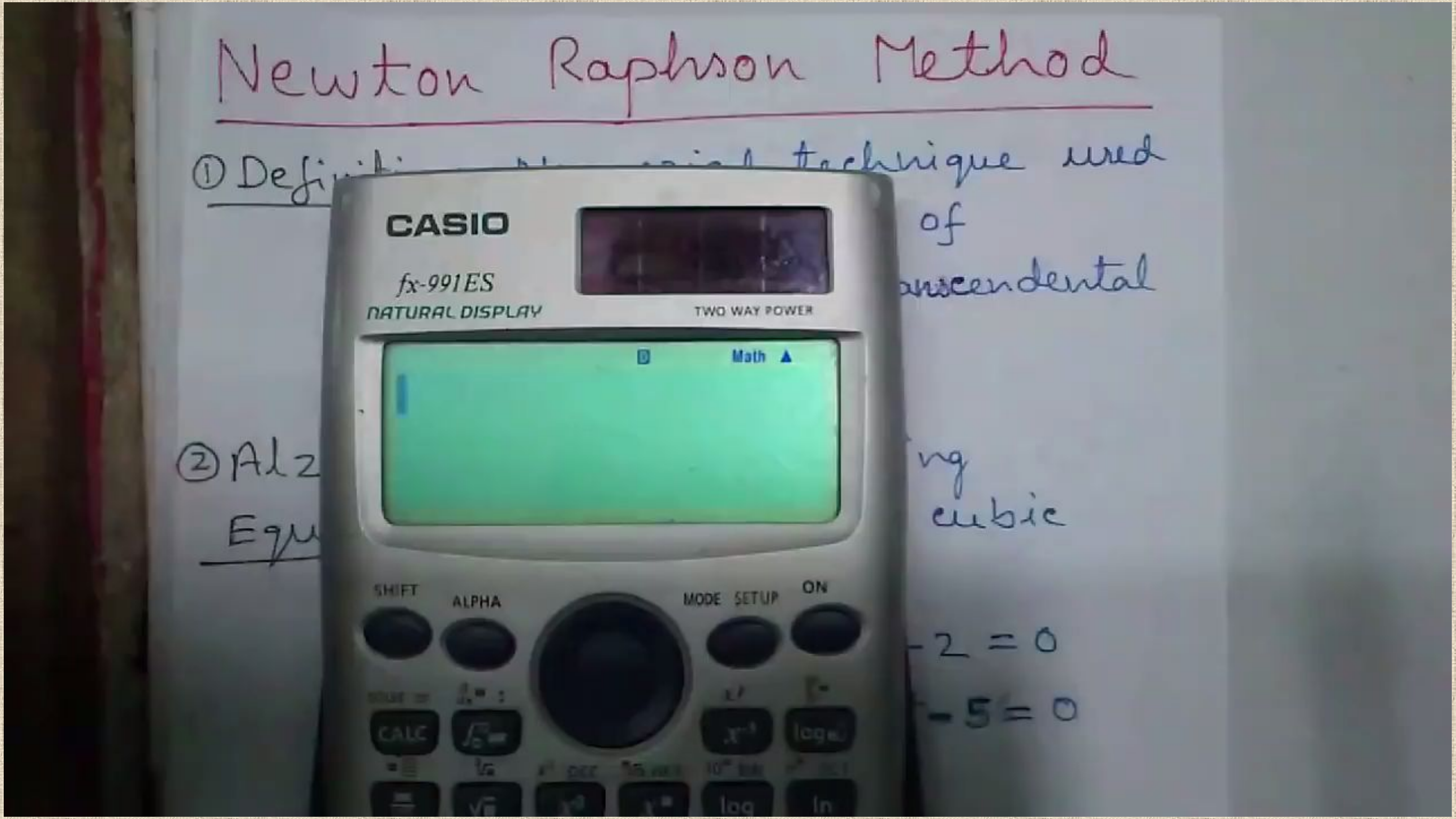
Since $x_n = x_{n-1}$ that is , $x_4 = x_3$ from the above table (slide no 10)

So the Newton - Rapshon method gives no new values of x

Therefore , the approximate root is correct to four decimal places.

Hence the require root is 2.2790 .

# How to solve Newton –Raphson Method by using Calculator



**Problem 1:** Find the root of the equation $x\sin(x)+\cos(x)=0$ ,using Newton-Rapshom method.

**Step 1**

**Solution :** $Given\ that,\ x\sin(x)+\cos(x)=0$

$$Let,\ f(x)=x\sin(x)+\cos(x)$$

$$\therefore f'(x)=\frac{d}{dx}x\sin(x)+\frac{d}{dx}\cos(x)$$

**x = u**

**sin (x)= v**

**Step 1 :**

$$f'(x)=\left[x\frac{d}{dx}\sin(x)+\sin(x)\frac{d}{dx}x\right]-\sin(x)$$

$$f'(x)=[x\cos(x)+\sin(x).1]-\sin(x)$$

$$\therefore f'(x)=x\cos(x)+\sin(x)-\sin(x)$$

$$=x\cos(x)$$

$$\frac{d}{dx}(uv)=u\frac{d}{dx}v+v\frac{d}{dx}u$$

$$\frac{d}{dx}\cos(x)=-\sin(x)$$

$$\frac{d}{dx}\sin(x)=\cos(x)$$

$$\frac{d}{dx}x=1$$

**Hints: Calculator must be in radian Mode.**

## Step 2 :

$$\because\ f(x)=x\sin(x)+\cos(x)$$

$$for,\ x=2,\ then\ f(2)=2\sin(2)+\cos(2)$$

$$=1.40>0$$

$$\because\ f(x)=x\sin(x)+\cos(x)$$

$$for,\ x=3,\ then\ f(3)=3\sin(3)+\cos(3)$$

$$=-0.56<0$$

Since f (2) and f (3) are of opposite sign so at least one real root lies between 2 and 3.

## Step 3 :

we know that from Newton-Rapshon method ,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$\because\ f(x)=x\sin(x)+\cos(x)$$

$$\therefore f(x_n)=x_n\sin(x_n)+\cos(x_n)$$

$$x_{n+1}=x_n-\frac{x_n\sin(x_n)+\cos(x_n)}{x_n\cos(x_n)}\ ......(1)$$

$$\because\ f'(x)=x\cos(x)$$

$$\therefore f'(x_n)=x_n\cos(x_n)$$

## Step 4 :

Now Putting n = 0 and Let , the initial value $x_0 = 2.79$ *in the above equation of* (1)

## Step 5 :

we are capable to find the successive improved approximations are as following table:

**Hints: Calculator must be in radian Mode.**

| No. of Iterations ,n | $x_n$ | $x_{n+1}=x_n-\dfrac{x_n\sin(x_n)+\cos(x_n)}{x_n\cos(x_n)}$ |
|---|---|---|
| 0 | $x_0=2.79\,(say)$ | $x_1=2.79-\dfrac{2.79\sin(2.79)+\cos(2.79)}{2.79\cos(2.79)}=2.7984$ |
| 1 | $x_1=2.7984$ | $x_2=2.7984-\dfrac{2.7984\sin(2.7984)+\cos(2.7984)}{2.7984\cos(2.7984)}=2.79834$ |

$$x_{0+1}=x_0-\frac{x_0\sin(x_0)+\cos(x_0)}{x_0\cos(x_0)}$$

$$x_1=2.79-\frac{2.79\sin(2.79)+\cos(2.79)}{2.79\cos(2.79)}$$

$$=2.7984$$

$$x_{1+1}=x_1-\frac{x_1\sin(x_1)+\cos(x_1)}{x_1\cos(x_1)}$$

$$x_2=x_1-\frac{x_1\sin(x_1)+\cos(x_1)}{x_1\cos(x_1)}$$

$$=2.7984-\frac{2.7984\sin(2.7984)+\cos(2.7984)}{2.7984\cos(2.7984)}$$

$$=2.79834$$

From the above table , we get ,

$$\therefore x_1 = 2.7984 \quad \text{and} \quad \therefore \ x_2 = 2.79834$$

$$\therefore \ x_2 \approx x_1$$

The approximate root is correct to three decimal places. Hence the require root is 2.7984.

You have to do mode on radian to your calculator

**Problem 02:** Find the real root of the equation $x^2 - 4\sin(x) = 0$ correct to four decimal places using Newton-Rapshon method. Solution

$$Solution:$$

$$Given\ that,\ x^2 - 4\sin(x) = 0$$

$$let,\ f(x) = x^2 - 4\sin(x)$$

$$\therefore f'(x) = \frac{d}{dx}x^2 - \frac{d}{dx}4\sin(x)$$

$$\therefore f'(x) = 2x - 4\cos(x)$$

$$\frac{d}{dx}x^n = n\,x^{n-1}$$

$$\therefore \frac{d}{dx}x^2 = 2\,x^{2-1} = 2x$$

$$\frac{d}{dx}a\,f(x) = a\frac{d}{dx}f(x) = a\,f'(x)$$

$$\therefore \frac{d}{dx}4\sin(x) = 4\frac{d}{dx}\sin(x) = 4\cos(x)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\because f(x)=x^2-4\sin(x)$$

$$for\ x=a=-1, then\ f(a)=a^2-4\sin(a)$$

$$\therefore f(-1)=(-1)^2-4\sin(-1)=-2.36<0$$

$$\because f(x)=x^2-4\sin(x)$$

$$for\ x=b=-2, then\ f(b)=b^2-4\sin(b)$$

$$\therefore f(-2)=(-2)^2-4\sin(-2)=0.36>0$$

Since f (-1) and f (-2) are of opposite sign , so at least one real root lies between -1 and -2.

we know that from Newton-Rapshon method ,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$\because f(x)=x^2-4\sin(x) \qquad \because f'(x)=2x-4\cos(x)$$

$$\therefore f(x_n)=x^2_n-4\sin(x_n) \qquad \therefore f'(x_n)=2x_n-4\cos(x_n)$$

$$x_{n+1}=x_n-\frac{x^2_n-4\sin(x_n)}{2x_n-4\cos(x_n)}........(1)$$

Now Putting n = 0 and Let , the initial value , $x_0 = -1.9$ *in the above equation of* (1)

we are capable to find the successive improved approximations are as following table:

| **No. of Iterations, n** | $x_{n+1}$ | $x_{n+1} = x_n - \frac{x^2_n - 4\sin(x_n)}{2x_n - 4\cos(x_n)}$ |
|---|---|---|
| $n=0$ | $x_{0+1} = x_1 = x_0 - \frac{x^2_0 - 4\sin(x_0)}{2x_0 - 4\cos(x_0)}$ | $x_1 = -1.9 - \frac{(-1.9)^2 - 4\sin(-1.9)}{2(-1.9) - 4\cos(-1.9)} = -1.93$ |
| $n=1$ | $x_{1+1} = x_2 = x_1 - \frac{x^2_1 - 4\sin(x_1)}{2x_1 - 4\cos(x_1)}$ | $x_2 = -1.93 - \frac{(-1.93)^2 - 4\sin(-1.93)}{2(-1.93) - 4\cos(-1.93)} = -1.9338$ |

*For* , $n=0$, *let* $x_0 = -1.9$

$$x_{0+1} = x_0 - \frac{x^2_0 - 4\sin(x_0)}{2x_0 - 4\cos(x_0)}$$

$$\therefore x_1 = -1.9 - \frac{(-1.9)^2 - 4\sin(-1.9)}{2(-1.9) - 4\cos(-1.9)} = -1.93$$

*For* , $n=1$, *and* $x_1 = -1.93$

$$x_{1+1} = x_1 - \frac{x^2_1 - 4\sin(x_1)}{2x_1 - 4\cos(x_1)}$$

$$\therefore x_2 = -1.93 - \frac{(-1.93)^2 - 4\sin(-1.93)}{2(-1.93) - 4\cos(-1.93)} = -1.9338$$

From the above table , we get ,

$x_1 = -1.93$ and $x_2 = -1.9338$

$$\therefore \ x_2 \approx x_1$$

The approximate root is correct to two decimal places.
Hence the require root is -1.93

## Algorithm for Newton-Raphson method:

| Steps | Task |
|---|---|
| 01 | Define f(x) |
| 02 | $Define\ , f'(x)$ |
| 02 | $\mathrm{Re}\,ad\ the\ initial\ value\ , x_0$ |
| 03 | Set , n =0 |
| 04 | n = n+1 |
| 05 | Calculate , $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ |

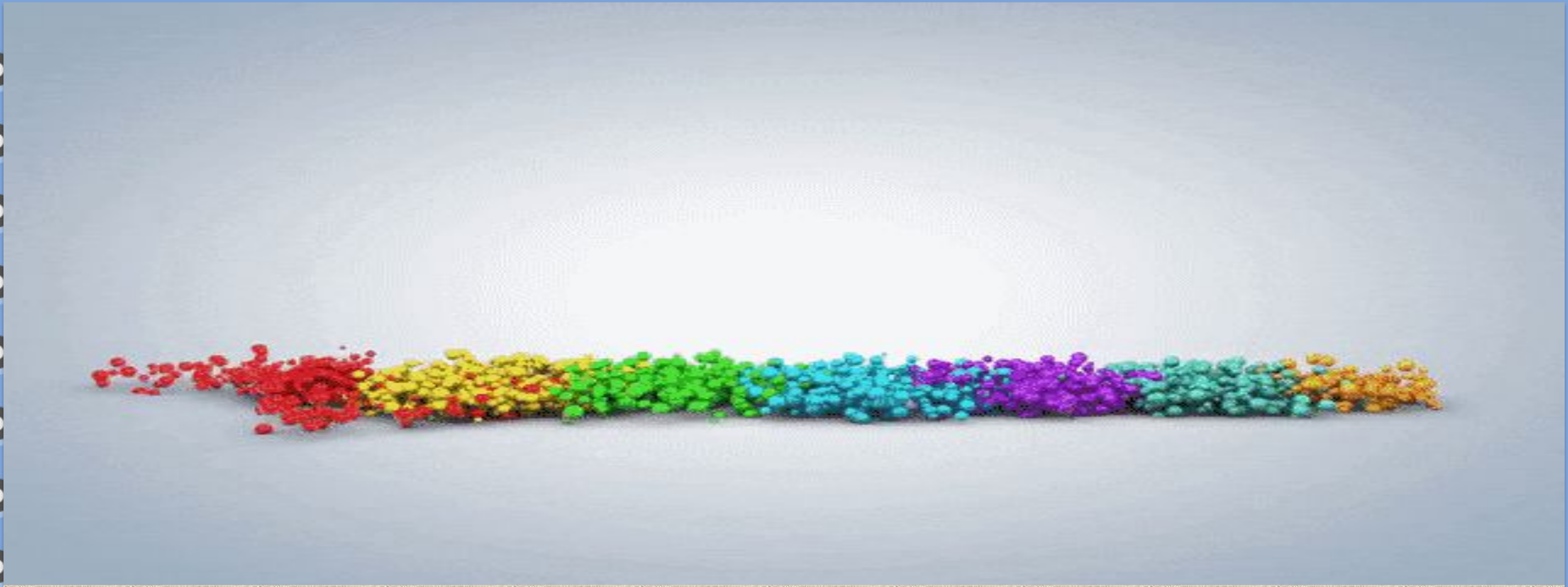
## Algorithm for Newton-Raphson method:

| Steps | Task |
| --- | --- |
| 06 | If $\lvert x_n - x_{n-1} \rvert \approx 0.0001$<br>$Then\ go\ to\ step\ 8$<br>$else\ if$<br>$n = n + 1$<br>$Go\ to\ step\ 5.$ |
| 07 | Print $x_n$ , the desired root |
| 08 | Stop |

# Practice Work

**Find the root of the following equation by using Newton – Raphson method :**

1. $2x = \ln x + 7$ *correct to 4 decimal places*
2. $3x + \sin x = e^{x}$ *correct to 3 decimal places.*
3. $\cos x = 3x - 1$ *correct to 3 decimal places.*
4. $e^{x} \tan x = 1$ correct to 3 decimal
5. $3x = \ln_{10} x + 7$ *correct to 4 decimal places*



# For Stay With Me