Interpolation

3.Lagrange Interpolation Formula

Types of Interpolation (According to Method)

Interpolation

Newton's Interpolation Formula

If the difference of value of x is equal

Lagrange's Interpolation Formula

If the difference of value of x is Unequal or Equal.

Lagrange's Interpolation Formula

Lagrange Polynomials

$$L_2(x) = f(x_1)\frac{x-x_2}{x_1-x_2}\cdot\frac{x-x_3}{x_1-x_3} + f(x_2)\frac{x-x_1}{x_2-x_1}\cdot\frac{x-x_3}{x_2-x_3} + f(x_3)\frac{x-x_1}{x_3-x_1}\cdot\frac{x-x_2}{x_3-x_2}$$

Given $(n+1)$ values of the function $f(x)$
for $x = x_0, x_1, x_2, \ldots.,,, x_n$
namely $f(x_0), f(x_1), \ldots, f(x_n)$ respectively,
the formula states

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)} f(x_2) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots . . \ldots \ldots \ldots \ldots \ldots \ldots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

Lagrange's Interpolation Formula

Given $(n+1)$ values of the function $f(x)$

for y= $y_0, y_1, y_2, \dots, y_n$

namely $f(y_0), f(y_1), \dots, f(y_n)$ respectively,

the formula states

$$f(y) = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)} f(y_0) + \frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)} f(y_1)$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)\dots(y-y_n)}{(y_2-y_0)(y_2-y_1)\dots(y_2-y_n)} f(y_2) + \dots \dots \dots$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})} f(y_n)$$

Lagrange's Interpolation Formula

Explain:

$(x - x_0)$

Part 1 : for x_0 = $\dfrac{(x - x_1)(x - x_2)(x - x_3)\ldots\ldots\ldots\ldots\ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)\ldots\ldots\ldots\ldots\ldots (x_0 - x_n)} f(x_0) / y_0$

$(x - x_1)$

Part 2 : for x_1 = $\dfrac{(x - x_0)(x - x_2)(x - x_3)\ldots\ldots\ldots\ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\ldots\ldots\ldots (x_1 - x_n)} f(x_1) / y_1$

$(x - x_n)$

Part last, n : for x_n = $\dfrac{(x - x_0)(x - x_2)(x - x_3)\ldots\ldots\ldots\ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\ldots\ldots\ldots (x_n - x_{n-1})} f(x_n) / y_n$

Problem 1: Apply Lagrange's formula to find $\log 656$ using the following values:

y

x_0	x_1	x_2	x_3

x	654	658	659	661
f(x) = y	2.8156	2.8182	2.8189	2.8202

$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

x

$\log 656$

Solution:

Here, $x_0 = 654, \quad x_1 = 658,$

$x_2 = 659, \quad x_3 = 661$ and $\mathrm{x} = 656$

$f(x_0) = 2.8156$, $f(x_1) = 2.8182$,

$f(x_2) = 2.8189$, $f(x_3) = 8202$

By Lagrange's formula, we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To find $\log 65$

Here, $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 661$ and x= 656

$f(x_0) = 2.8156$, $f(x_1) = 2.8182$, $f(x_2) = 2.8189$,
$f(x_3) = 8202$

Substituting the values of x_0, x_1, x_2 and x_3 and corresponding values of $f(x)$ in the above equation, we get

$$\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$

$x_0 = 654$

$x_1 = 658$

$x_2 = 659$

$$= \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156$$

$$= 0.60334$$

$x_3 = 661$

$x = 656$

$f(x_0) = 2.8156$

$$\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$x = 656$$

$$x_0 = 654$$

$$= \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$

$$= 7.0455$$

$$x_1 = 658$$

$$x_2 = 659,$$

$$x_3 = 661$$

$$f(x_1) = 2.8182$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$= \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(654-661)} \times 2.8189$$

$$= -5.6378$$

$$x_0 = 654$$

$$x_1 = 658$$

$$x_2 = 659$$

$$x_3 = 661$$

$$f(x_2) = 2.8189$$

$$\mathrm{x} = 656$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$= \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$= 0.80577$$

$x_0 = 654$

$x_1 = 658$

$x_2 = 659$

$x_3 = 661$

$\mathrm{x} = 656$

From (1) we get,

$f(x_3) = 2.8202$

$$f(656) = 0.60334 + 7.0455 - 5.6378 + 0.80577 = 2.81681$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$= 0.80577$$

$\mathrm{x} = 656$

$x_0 = 654$

$x_1 = 658$

$x_2 = 659$

$x_3 = 661$

$f(x_3) = 2.8202$

Substituting the values of x_0, x_1, x_2, x_3 and corresponding values of f(x) in the above equation, we get

$$\therefore \log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156$$

$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182$$

$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189$$

$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2.8202$$

$$\Rightarrow \log_{10} 656 = \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \times 2.8156 + \frac{2(-3)(-5)}{4(-1)(-3)} \times 2.8182$$

$$+ \frac{2.(-2)(-5)}{5.1.(-2)} \times 2.8189 + \frac{2(-2)(-3)}{7.\,3.\,2} \times 2.8202$$

$$= 0.60334 + 7.0455 - 5.6378 + 0.80577$$

$$\therefore \log_{10} 656 = 2.81681. \text{ Ans}$$

Problem 2: Apply Lagrange's formula to find the value of y at x = 27 using the following values:

x	22	23.5	25.2	28.7
y	2.8	3.5	4.6	5.3

Here , x = 27 , $x_0 = 22$, $x_1 = 23.5$, $x_2 = 25.2$, $x_3 = 28.7$

$y_0 = 2.8$, $y_1 = 3.5$, $y_2 = 4.6$, $y_3 = 5.3$

By Lagrange's formula, we have

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$f(x) = \frac{(27-23.5)(27-25.2)(27-28.7)}{(22-23.5)(22-25.2)(22-28.7)} \times 2.8$$

Here , x = 27 ,

$x_0 = 22$, $y_0 = 2.8$,

$$+ \frac{(27-22)(27-25.2)(27-28.7)}{(23.5-22)(23.5-25.2)(23.5-28.7)} \times 3.5$$

$x_1 = 23.5$, $y_1 = 3.5$,

$x_2 = 25.2$, $y_2 = 4.6$,

$$+ \frac{(27-22)(27-23.5)(27-28.7)}{(25.2-27)(25.2-23.5)(25.2-28.7)} \times 4.6$$

$x_3 = 28.7$,

$y_3 = 5.3$

$$+ \frac{(27-22)(27-23.5)(27-25.2)}{(28.7-22)(28.7-23.5)(28.7-25.2)} \times 5.3$$

$$f(x) = \frac{(27-23.5)(27-25.2)(27-28.7)}{(22-23.5)(22-25.2)(22-28.7)} \times 2.8$$

$$+ \frac{(27-22)(27-25.2)(27-28.7)}{(23.5-22)(23.5-25.2)(23.5-28.7)} \times 3.5$$

$$+ \frac{(27-22)(27-23.5)(27-28.7)}{(25.2-27)(25.2-23.5)(25.2-28.7)} \times 4.6$$

$$+ \frac{(27-22)(27-23.5)(27-25.2)}{(28.7-22)(28.7-23.5)(28.7-25.2)} \times 5.$$

Here , x = 27 ,

$x_0 = 22$, $y_0 = 2.8$,

$x_1 = 23.5$, $y_1 = 3.5$,

$x_2 = 25.2$, $y_2 = 4.6$,

$x_3 = 28.7$, $y_3 = 5.3$

$$f(27) = \frac{(3.5)(1.8)(-1.7)}{(-1.5)(-3.2)(-6.7)} \times 2.8 + \frac{5(1.8)(-1.7)}{(1.5)(-1.7)(-5.2)} \times 3.5$$

$$+ \frac{(5)(3.5)(-1.7)}{(3.2)(1.7)(-3.5)} \times 4.6 + \frac{(5)(3.5)(1.8)}{(6.7)(5.2)(3.5)} \times 5.$$

$= 0.93246 - 4.03846 + 7.18750 + 1.36912$

$= 5.45062$

$$\therefore f(27) = 5.45062$$

Here There is no fixed value of x.

Problem 3: Given the following values:

x	-1	0	2	5
y	9	5	3	15

Find the Lagrange's polynomial .

Solution : By Lagrange's formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$x_0 = -1$	$y_0 = 9$
$x_1 = 0$	$y_1 = 5$
$x_2 = 2$	$y_2 = 3$
$x_3 = 5$	$y_3 = 15$

x	-1	0	2	5
y	9	5	3	15

$$f(x) = \frac{(x-0)(x-2)(x-5)}{(-1-0)(-1-2)(-1-5)} \times 9 + \frac{(x+1)(x-2)(x-5)}{(0+1)(0-2)(0-5)} \times 5$$

$$+ \frac{(x+1)(x-0)(x-5)}{(2+1)(2-0)(2-5)} \times 3 + \frac{(x+1)(x-0)(x-2)}{(5+1)(5-0)(5-2)} \times 15$$

$$= \frac{(x)(x-2)(x-5)}{(-1)(-3)(-6)} \times 9 + \frac{(x+1)(x-2)(x-5)}{(1)(-2)(-5)} \times 5$$

$$+ \frac{(x+1)(x)(x-5)}{(3)(2)(-3)} \times 3 + \frac{(x+1)(x)(x-2)}{(6)(5)(3)} \times 15$$

$$= \frac{x^3-7x^2+10x}{(-2)} + \frac{x^3-6x^2+3x+10}{2} + \frac{x^3-4x^2-5x}{(-6)} + \frac{x^3-x^2-2x}{(6)}$$

$$= \frac{3(x^3-7x^2+10x)+3(x^3-6x^2+3x+10)-(x^3-4x^2-5x)+(x^3-x^2-2x)}{6}$$

$$= \frac{-3x^3+21x^2-30x+3x^3-18x^2+9x+30-x^3+4x^2+5x+x^3-x^2-2x}{6}$$

$$= \frac{6x^2-18x+30}{6} = x^2 - 3x + 5$$

Derive a Lagrange's Interpolation formula for degree 3.

Solution:

Let the function be $y = f(x)$.

For degree 3 we choose the value of x are x_0, x_1, x_2

and the corresponding values of the function $f(x)$ be

$f(x_0)$, $f(x_1)$, and $f(x_2)$.

Now we fit the polynomial of $f(x)$ of degree 3.

Let, $f(x) = a_0(x - x_1)(x - x_2) + a_1(x - x_0)(x - x_2)$

$$+a_2(x - x_0)(x - x_1)\ldots\ldots(1)$$

To find the values of a_0, a_1 and a_2

we put $x = x_0, \; x_1, \; x_2$ respectively in (1), we get

$when\, x = x_0$

$$f(x) = a_0(x - x_1)(x - x_2) + a_1(x - x_0)(x - x_2) + a_2(x - x_0)(x - x_1) \ldots\ldots (1)$$

$$\therefore f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) + a_1(x_0 - x_0)(x_0 - x_2) + a_2(x_0 - x_0)(x - x_1)$$

$$\therefore f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2)$$

$$\therefore a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$when\ x = x_1$$

$$f(x) = a_0(x - x_1)(x - x_2) + a_1(x - x_0)(x - x_2) + a_2(x - x_0)(x - x_1) \ldots\ldots(1)$$

$$\therefore f(x_1) = a_0 (x_1 - x_1)(x_1 - x_2) + a_1(x_1 - x_0)(x_1 - x_2) + a_2(x_1 - x_0)(x_1 - x_1)$$

$$\therefore f(x_1) = a_1 (x_1 - x_0)(x_1 - x_2)$$

$$\therefore a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)}$$

Similarly , $a_2 = \dfrac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$

Substituting these values a_0, a_1 and a_2 in (1) we get

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

This is the required Lagrange's interpolation formula fro degree 3.

Problem 4: Given the following values:

x	50	52	54	56
	3.684	3.732	3.779	3.825

use the Lagrange's formula to Find the value of x when y= $\sqrt[3]{x} = 3.756$

•

Given , y= $\sqrt[3]{x} = 3.756$ and

$X_0 = 50$, $y_0 = 3.684$,

$x_1 = 52$, $y_1 = 3.732$,

$x_2 = 54$, $y_2 = 3.779$,

$x_3 = 56$, $y_3 = 3.825$

By Lagrange's formula, we have

$$\mathbf{x} = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3 \ldots\ldots(1)$$

Given , y= $\sqrt[3]{x} = 3.756$ and

$X_0 = 50$, $x_1 = 52$, $x_2 = 54$, $x_3 = 56$,

$y_0 = 3.684$, $y_1 = 3.732$, $y_2 = 3.779$, $y_3 = 3.825$

$$x = \frac{(3.756-3.732)(3.756-3.779)(3.756-3.825)}{(3.684-3.732)(3.684-3.779)(3.684-3.825)} \times 50$$

$$+ \frac{(3.756-3.684)(3.756-3.779)(3.756-3.825)}{(3.732-3.684)(3.732-3.779)(3.732-3.825)} \times 52$$

$$+ \frac{(3.756-3.684)(3.756-3.732)(3.756-3.825)}{(3.779-3.684)(3.779-3.732)(3.779-3.825)} \times 54$$

$$+ \frac{(3.756-3.684)(3.756-3.732)(3.756-3.779)}{(3.825-3.684)(3.825-3.732)(3.825-3.779)} \times 56 = 52.9879$$

For Practice:

1. Estimate $\sqrt{155}$ using Lagrange's inter pollution's formula from the table given below :

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

2. Apply Lagrange's interpolation formula to find the values of f(8) and f(15) from the following table :

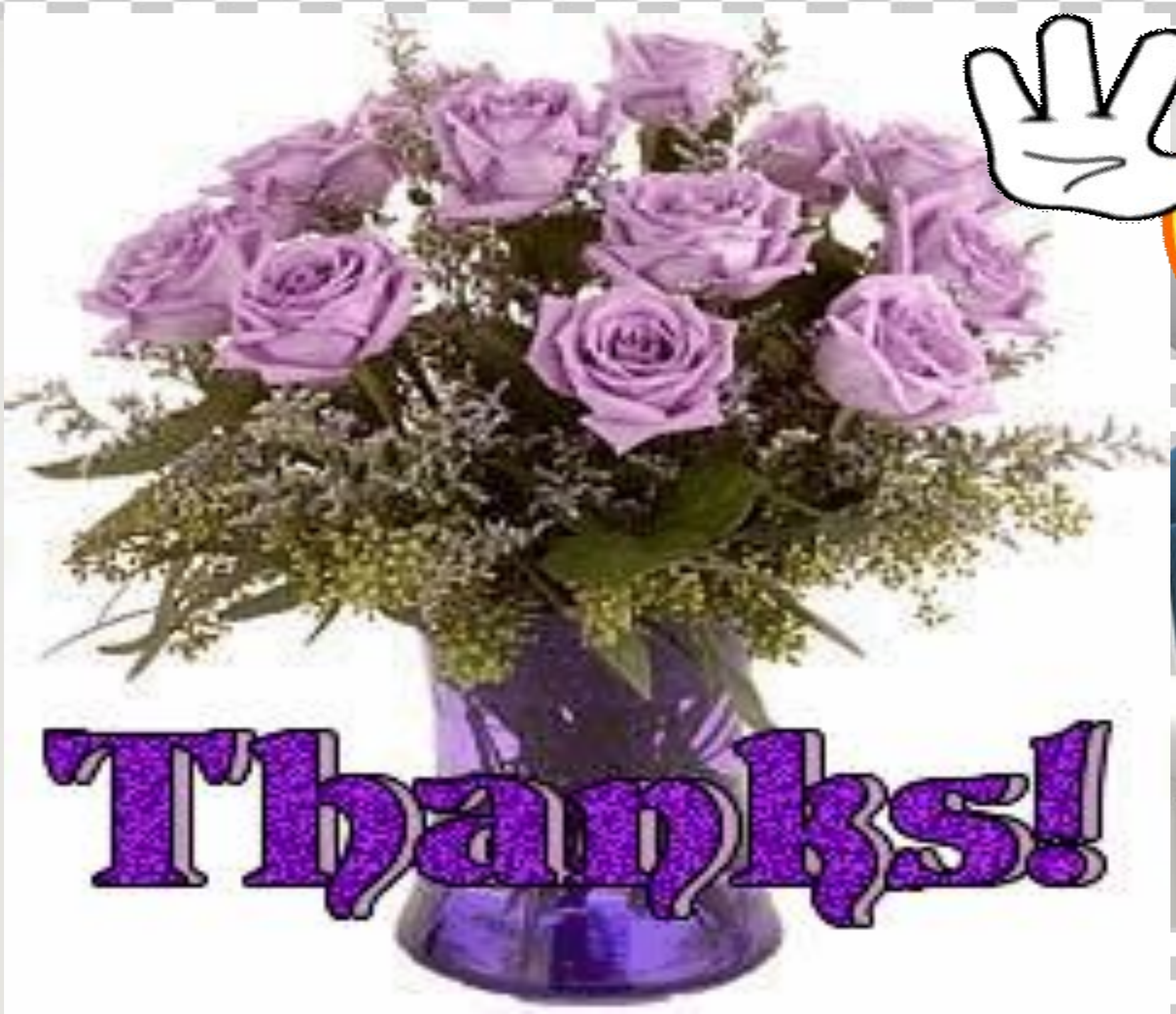
x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Ans. 448, 3150

3. Apply Lagrange's interpolation formula to find the value of y when x = 10

x	5	6	9	11
y	12	13	14	16

Ans. 14.66


Thanks!