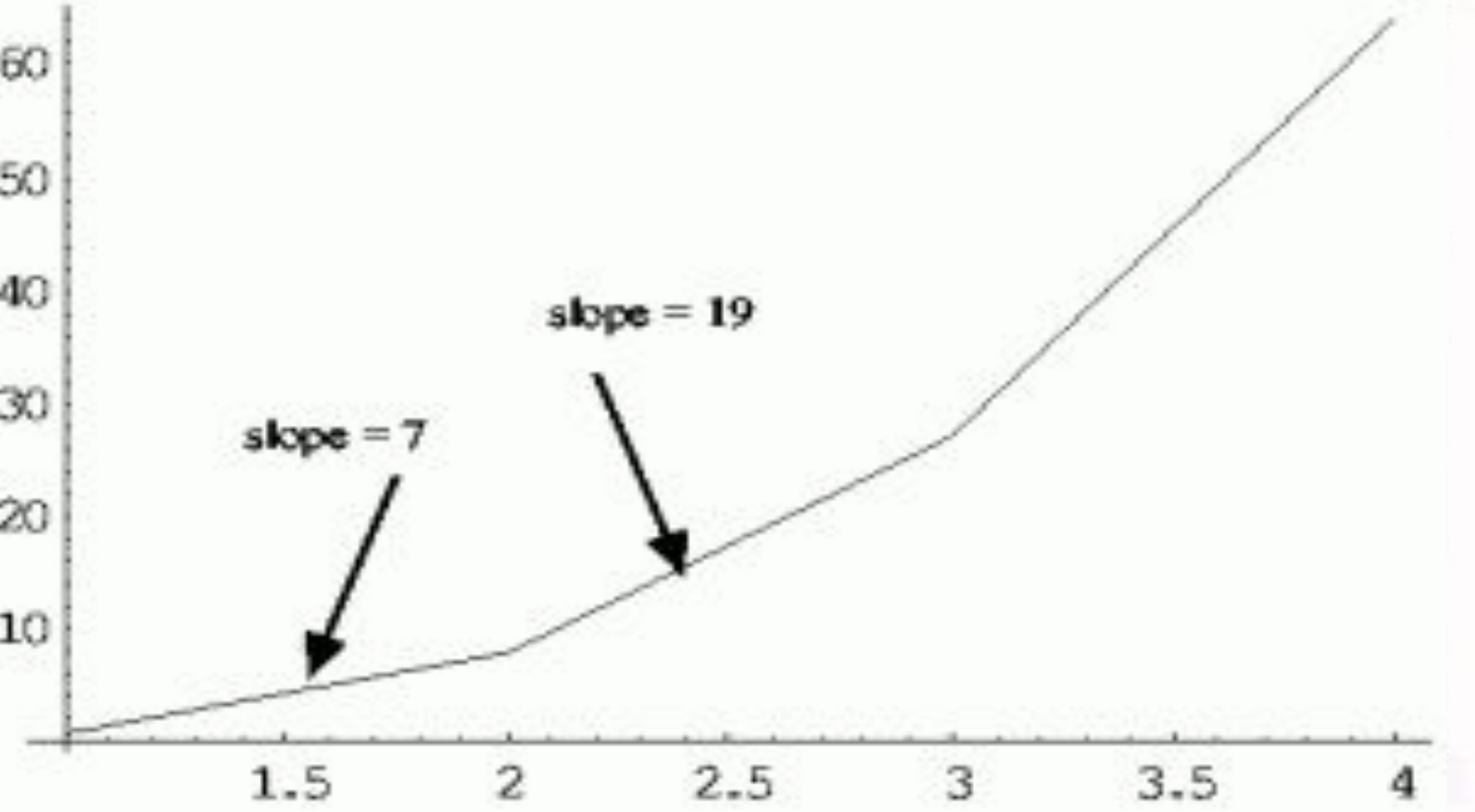
60
50
40
30
20
10
1.5
2
2.5
3
3.5
4
slope = 7
slope = 19

Numerical Differentiation

Chapter 6

Learning results

1. Derive the Numerical Derivative For Newton's Forward Difference Formula

2. How to solve the Problem of Numerical Derivative For Newton's Forward Difference Formula

Numerical Derivative For Newton's Forward Difference Formula

Basic Concept:

$$\frac{dy}{dx} \rightarrow \text{Derivative}$$

The differential coefficient

Or

The dependent variable y with respect to independent variable x

Example

$$y = 3x^2 - 2x + 1$$

$$\frac{dy}{dx} = ?$$

Consider, Newton's Forward Difference formula

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \ldots\ldots\ldots\ldots\ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0$$

where, $u = \frac{x - x_0}{h}$; h is the difference of x values in the chart.

Here y is dependent variable and u is independent variable

Since, $u = \frac{x - x_0}{h}$

$$\frac{d}{dx} c\, f(x) = c \frac{d}{dx} f(x)$$

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x - x_0}{h}\right)$$

$$\frac{d}{dx}(u + v) = \frac{d}{dx} u + \frac{d}{dx} v$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{h}\frac{d}{dx}(x - x_0)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{h}\left[\frac{d}{dx}(x) - \frac{d}{dx}(x_0)\right]$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{h}\left[\frac{d}{dx}(x) - \frac{d}{dx}(x_0)\right]$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{h}[1 - 0]$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{h} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \ldots\ldots\ldots\ldots\ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0$$

$$\frac{dy}{du} = \frac{d}{du}\Big(y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 + \ldots + n \text{ terms}\Big)$$

$$\Rightarrow \frac{dy}{du} = 0 + 1.\Delta y_0 + \frac{\Delta^2 y_0}{2}\frac{d}{du}(u^2 - u)$$

$$+ \frac{1}{3!}\Delta^3 y_0 \frac{d}{du}(u^3 - 3u^2 + 2u)$$

$$+ \frac{1}{4!}\Delta^4 y_0 \frac{d}{du}(u^4 - 6u^3 + 11u^2 - 6u)$$

$$+ \frac{1}{5!}\Delta^5 y_0 \frac{d}{du}[u^5 - 10u^4 + 35u^3 - 50u^2 + 24u + \dots\dots\dots + (n \text{ terms})]$$

$$\Rightarrow \frac{dy}{du} = 0 + 1.\Delta y_0 + \frac{\Delta^2 y_0}{2}\left[\frac{d}{du}(u^2) - \frac{d}{du}(u)\right]$$

$$+ \frac{1}{3!}\Delta^3 y_0\left[\frac{d}{du}\left(u^3\right) - \frac{d}{du}3u^2 + \frac{d}{du}2u\right)\,]$$

$$+ \frac{1}{4!}\Delta^4 y_0\left[\frac{d}{du}(u^4) - \frac{d}{du}6u^3 + \frac{d}{du}11u^2 - \frac{d}{du}6u)\right]$$

$$+ \frac{1}{5!}\Delta^5 y_0\left[\frac{d}{du}u^5 - \frac{d}{du}10u^4 + \frac{d}{du}35u^3 - \frac{d}{du}50u^2 + \frac{d}{du}24u\right] + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (n\ terms)]$$

$$\Rightarrow \frac{dy}{du} = \Delta y_0 + \frac{\Delta^2 y_0}{2}[2u - 1]$$

$$+ \frac{1}{6}\Delta^3 y_0[3u^2 - 6u + 2]$$

$$+ \frac{1}{24}\Delta^4 y_0[4u^3 - 18u^2 + 22u - 6]$$

$$+ \frac{1}{120}\Delta^5 y_0 [5u^4 - 40u^3 + 105u^2 - 100u + 24] + \ldots\ldots + (n\ terms)]$$

$$\frac{d}{dx} x^n = \mathrm{n}\ x^{n-1}$$

$$\Rightarrow \frac{dy}{du} = \Delta y_0 + \frac{1}{2}[2u - 1]\,\Delta^2 y_0$$

$$+ \frac{1}{6}\,[3u^2 - 6u + 2\,]\,\Delta^3 y_0$$

$$+ \frac{1}{24}\,[4u^3 - 18u^2 + 22u - 6\,]\,\Delta^4 y_0$$

$$+ \frac{1}{120}\,[5u^4 - 40u^3 + 105u^2 - 100u + 24]\Delta^5 y_0 + \ldots\ldots + (n\ terms)]\ldots\ldots\ldots\ldots(2)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{h} \cdot [\ \Delta y_0 + \frac{1}{2}(2u - 1)\Delta^2 y_0$$

$$+ \frac{1}{6}(3u^2 - 6u + 2)\ \Delta^3 y_0$$

$$+ \frac{1}{24}(4u^3 - 18u^2 + 22u - 6)\ \Delta^4 y_0$$

$$+ \frac{1}{120}[5u^4 - 40u^3 + 105u^2 - 100u + 24]\Delta^5 y_0 \ \ldots\ldots + (n\ terms)\ ]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

This formula can be used for computing the value of *dy/dx* for *non-tabular values* of x. For tabular values of x, the formula takes a simpler form, for

by setting $x = x_0$ we obtain $u = 0$ in Equation (3)

$$\frac{dy}{dx} = \frac{1}{h} \cdot \Big[\ \Delta y_0 + \frac{1}{2}(2 \times 0 - 1)\Delta^2 y_0$$

$$+ \frac{1}{6}\ (3 \times 0 - 6 \times 0 + 2\ )\ \Delta^3 y_0$$

$$+ \frac{1}{24}\ (4 \times 0 - 18 \times 0 + 22 \times 0 - 6\ )\ \Delta^4 y_0$$

$$+ \frac{1}{120}\ [5 \times 0 - 40 \times 0 + 105 \times 0 - 100 \times 0 + 24]\Delta^5 y_0\ \ldots\ldots + (n\ terms)\ \Big]$$

$$\frac{dy}{dx} = \frac{1}{h} . [\ \Delta y_0 + \frac{1}{2}(-1)\Delta^2 y_0 + \frac{1}{6}\ (2\ )\ \Delta^3 y_0$$

$$+ \frac{1}{24}\ (-6\ )\ \Delta^4 y_0 + \frac{1}{120}\ [\,24]\Delta^5 y_0 + \ldots\ldots + (n\ terms\ )\ ]$$

$$\frac{dy}{dx} = \frac{1}{h} . [\ \Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\ \Delta^4 y_0 + \frac{1}{5}\ \Delta^5 y_0$$

$$+ \ldots\ldots\ \ldots\ldots\ldots\ldots + (n\ terms\ )\ ]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} \quad \therefore \quad \frac{d^2y}{dx^2} = \frac{d}{du}\frac{du}{dx}\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{du}\frac{1}{h}\frac{1}{h}\Big[\ \Delta y_0 + \frac{1}{2}(2u-1)\Delta^2 y_0$$

$$+\frac{1}{6}(3u^2 - 6u + 2)\Delta^3 y_0$$

$$+\frac{1}{24}(4u^3 - 18u^2 + 22u - 6)\Delta^4 y_0$$

$$+\frac{1}{120}[5u^4 - 40u^3 + 105u^2 - 100u + 24]\Delta^5 y_0 \ldots\ldots + (n \text{ terms})\Big]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\frac{d}{du}\left[\ \Delta y_0 + \frac{1}{2}(2u-1)\Delta^2 y_0 + \frac{1}{6}(3u^2 - 6u + 2)\Delta^3 y_0 + \frac{1}{24}(4u^3 - 18u^2 + 22u - 6)\,\Delta^4 y_0 + \frac{1}{120}[5u^4 - 40u^3 + 105u^2 - 100u + 24]\Delta^5 y_0 \ldots\ldots + (n\ terms)\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\Big[\ \mathbf{0} + \frac{1}{2}(2 \times 1 - 0)\Delta^2 y_0$$

$$+ \frac{1}{6}\ (6u - 6 \times 1 + 0)\Delta^3 y_0$$

$$+ \frac{1}{24}\ (12u^2 - 36u + 22 \times 1 - 0\ )\ \Delta^4 y_0$$

$$+ \frac{1}{120}\ [20u^3 - 120u^2 + 210u - 100 \times 1 + 0]\Delta^5 y_0 \ldots\ldots + (n\ terms)\Big]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\ \frac{1}{2}(2)\Delta^2 y_0 + \frac{1}{6}(6u - 6)\Delta^3 y_0 + \frac{1}{24}(12u^2 - 36u + 22)\ \Delta^4 y_0 + \frac{1}{120}[20u^3 - 120u^2 + 210u - 100]\Delta^5 y_0 \ldots\ldots + (n\ terms)\ \right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\Big[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{1}{24}(12u^2 - 36u + 22)\,\Delta^4 y_0 + \frac{1}{120}[20u^3 - 120u^2 + 210u - 100]\Delta^5 y_0 + \ldots\ldots + (n\ terms)\Big] \ldots\ldots\ldots\ldots(4)$$

by setting $x = x_0$ we obtain $u = 0$ in Equation (4)

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\Big[\Delta^2 y_0 + (0-1)\Delta^3 y_0 + \frac{1}{24}(12\times 0 - 36\times 0 + 22)\,\Delta^4 y_0 + \frac{1}{120}[20\times 0 - 120\times 0 + 210\times 0 - 100]\Delta^5 y_0 + \ldots\ldots + (n\ terms)\Big]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{1}{24}(22)\,\Delta^4 y_0 + \frac{1}{120}[-100]\Delta^5 y_0 + \ldots\ldots + (n\ terms)\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\,\Delta^4 y_0 - \frac{5}{6}\,\Delta^5 y_0 + \ldots\ldots + (n\ terms)\right]$$

Similarly ,

Newton's backward difference formula gives

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right) \quad \text{...........(5)}$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right). \quad \text{...........(6)}$$

Problem

Example- The values of x and y are given in the following table :-

x	0	1	2	3	4	5
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 1$.

Solution : Since the derivatives are required at $x = 1$, which is beginning of the table, therefore we shall use the first and second derivatives of Newtor's forward formula at $x = x_0$. The difference table is :

x	y					
0	6.9897					
		0.4139				
1	7.4036		-0.036			
		0.3779		0.0057		
2	7.7815		- 0.0303		- 0.0011	
		0.3476		0.0046		- 0.0001
3	8.1291		- 0.0257		- 0.0012	
		0.3219		0.0034		
4	8.8510		- 0.0223			
		0.2996				
5	8.7506					

Here h = 1 and x_0 = 1, the first derivative of Newton's forward formula at $x = x_0$ is :

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h} \cdot \left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \frac{1}{5} \Delta^5 y_0 + \ldots \ldots + (n \text{ terms}) \right] \ldots\ldots\ldots (1)$$

and the second derivative of Newton's forward formula at $x = x_0$ is

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \cdots\right] \ldots\ldots (2)$$

Putting $x_0 = 1$, $h = 1$ and corresponding values of the differences in (1) and (2), we get

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{1}\left[0.3779 - \frac{1}{2} \times (-0.0303) + \frac{1}{3}(0.0046) - \frac{1}{4}(-0.0012)\right]$$

$$= 0.39488$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{1}{(1)^2}\left[-0.0303 - 0.0046 + \frac{11}{12}(-0.0012)\right]$$

$$= -0.036$$

Problem

Example-☺. From the following table of values of x and y find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when x = 5.5 :

x	4.5	5	5.5	6	6.5	7.0	7.5
y	9.69	12.90	16.71	21.18	26.36	32.34	39.15

Solution : Since x = 5.5 towards the beginning of the table, we use Newton forward difference formula. The difference table is given below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
4.5	9.69						
		3.21					
5.0	12.90		0.6				
		3.81		0.06			
5.5	**16.71**		0.66		–0.01		
		4.47		0.05		0.05	
6.0	21.18		**0.71**		0.04		–0.15
		5.18		**0.09**		–0.1	
6.5	26.36		0.8		**–0.06**		
		5.98		0.03			
7.0	32.34		0.83				
		6.81					
7.5	39.15						

Here $h = 0.5$ and $x_0 = 5.5$

the first derivative of Newton's forward formula at $x = x_0$ is

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right]$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=5.5} = \frac{1}{0.5}\left[4.47 - \frac{1}{2}\times 0.71 + \frac{1}{3}\times 0.09 - \frac{1}{4}(-0.06)\right]$$

$$= 8.32$$

and the second derivative of Newton's forward formula at $x = x_0$ is

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{x=5.5} = \frac{1}{(0.5)^2}\left[0.71 - 0.09 + \frac{11}{12}(-0.06)\right]$$

$$= 2.26$$

Practice Work

Given that

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (i) x = 1.1

Ans. (i) 3.941; –3.3167;

From the following table find the first and second derivatives of sin x at (i) $x = 0^0$ (ii) $x = 40^0$ (iii) $x = 20^0$

x^0 :	0	10	20	30	40
$y = \sin x^0$:	0.000	0.1736	0.3420	0.5000	0.6428


Thank You