Numerical Differentiation

3rd Part : Runge- kutta of 4th order

Runge-Kutta Method of 4th order

Consider $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

The Runge-Kutta method of 4th order is given by

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

Problem

Apply Runge-Kutta method of fourth order to find an approximate value of y(0.1) and y(0.2) of $\frac{dy}{dx} = x + y^2$, y(0)=1 correct to three decimal places.

Solution :

Given data: $f(x, y) = x + y^2$, h=0.1

$x_0 = 0$ $\quad x_1 = 0.1$ $\quad x_2 = 0.2$

$y_0 = 1$ $\quad y_1 = ?$ $\quad y_2 = ?$

The Runge-Kutta method of 4th order is given by

$$
\left.
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \\
\text{where } k_1 &= hf(x_n, y_n) \\
k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 &= hf(x_n + h, y_n + k_3)
\end{aligned}
\right\} \text{------------------------}(1)
$$

Put n=0 in Eqn(1)

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad \text{------------------------}(2)$$

$$\left.\begin{aligned}
&\text{where } k_1 = hf(x_0, y_0) \\
&k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&k_4 = hf(x_0 + h, y_0 + k_3)
\end{aligned}\right\}$$

$$k_1 = hf(x_0, y_0)$$

$$= h\left[x_0 + (y_0)^2\right]$$

$$= 0.1\left[0 + (1)^2\right] = 0.1$$

$$\frac{dy}{dx} = f(x, y)$$

$$\frac{dy}{dx} = x + y^2,$$

$$x_0 = 0$$

$$y_0 = 1$$

$$h = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right]$$

$$= 0.1152$$

$$\frac{dy}{dx} = x + y^2,$$

$$x_0 = 0$$
$$y_0 = 1$$

$$h = 0.1$$

$$k_1 = 0.1$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1152}{2}\right)^2\right]$$

$$= 0.1168$$

$$\frac{dy}{dx} = x + y^2,$$

$$x_0 = 0$$

$$y_0 = 1$$

$$h = 0.1$$

$$k_2 = 0.1152$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= h\left[(x_0 + h) + (y_0 + k_3)^2\right]$$

$$= 0.1\left[(0 + 0.1) + (1 + 0.1168)^2\right]$$

$$= 0.1347$$

$$\frac{dy}{dx} = x + y^2$$

$$x_0 = 0$$

$$y_0 = 1$$

$$h = 0.1$$

$$k_3 = 0.1168$$

Substituting all these values in Eqn(2), we get

$$y_1 = 1 + \frac{1}{6}[0.1 + 2(0.1152) + 2(0.1168) + 0.1347] = 1.1164$$

Put n=1 in Eqn(1)

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \quad \text{-----------(3)}$$

Where

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$k_1 = hf(x_1, y_1)$$

$$= h\left[x_1 + (y_1)^2\right]$$

$$= 0.1\left[(0.1) + (1.1164)^2\right]$$

$$= 0.1346$$

$$\frac{dy}{dx} = x + y^2,$$

$h=0.1$

$x_1 = 0.1$

$y_1 = 1.1164$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{k_1}{2}\right)^2\right]$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1164 + \frac{0.1346}{2}\right)^2\right]$$

$$= 0.1551$$

$$\frac{dy}{dx} = x + y^2,$$

$h=0.1$

$x_1 = 0.1$

$y_1 = 1.1164$

$k_1 = 0.1346$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= h\left[\left(x_1 + \frac{h}{2}\right) + \left(y_1 + \frac{k_2}{2}\right)^2\right]$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.1164 + \frac{0.1551}{2}\right)^2\right]$$

$$= 0.1575$$

$$\frac{dy}{dx} = x + y^2,$$

$h = 0.1$

$x_1 = 0.1$

$y_1 = 1.1164$

$k_2 = 0.1551$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= h\left[(x_1 + h) + (y_1 + k_3)^2\right]$$

$$= 0.1\left[(0.1 + 0.1) + (1.1164 + 0.1575)^2\right]$$

$$= 0.1822$$

$$\frac{dy}{dx} = x + y^2,$$

$h=0.1$

$x_1 = 0.1$

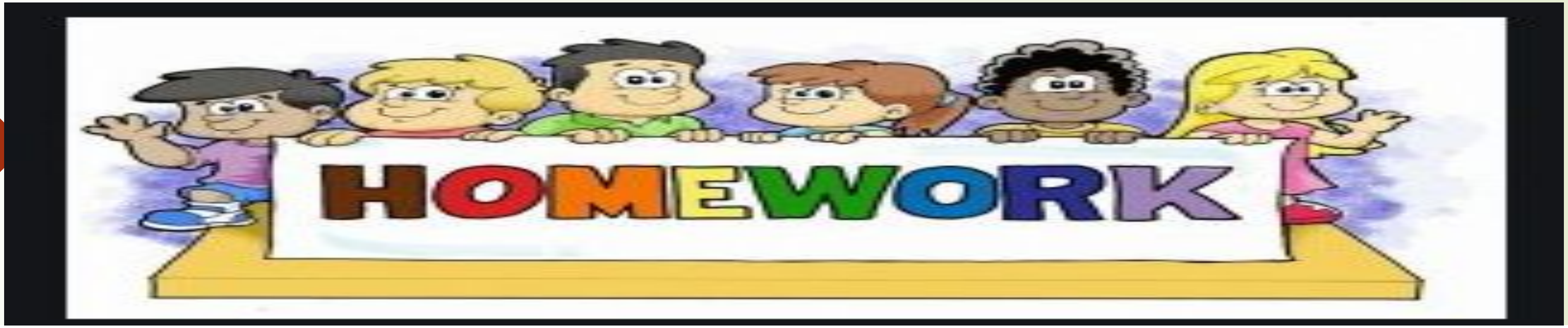
$y_1 = 1.1164$

$k_3 = 0.1575$

Substituting all these values in Eqn(3), we get

$$y_2 = 1.1164 + \frac{1}{6}[0.1346 + 2(0.1551) + 2(0.1575) + 0.1822]$$

$$= 1.2734$$



Solve $y' = 1 + y^2$ with $y(0) = 0$ for $x = 0.2\ (0.2)\ 0.6$ by Runge-Kutta method of fourth order. [NUH-1999]

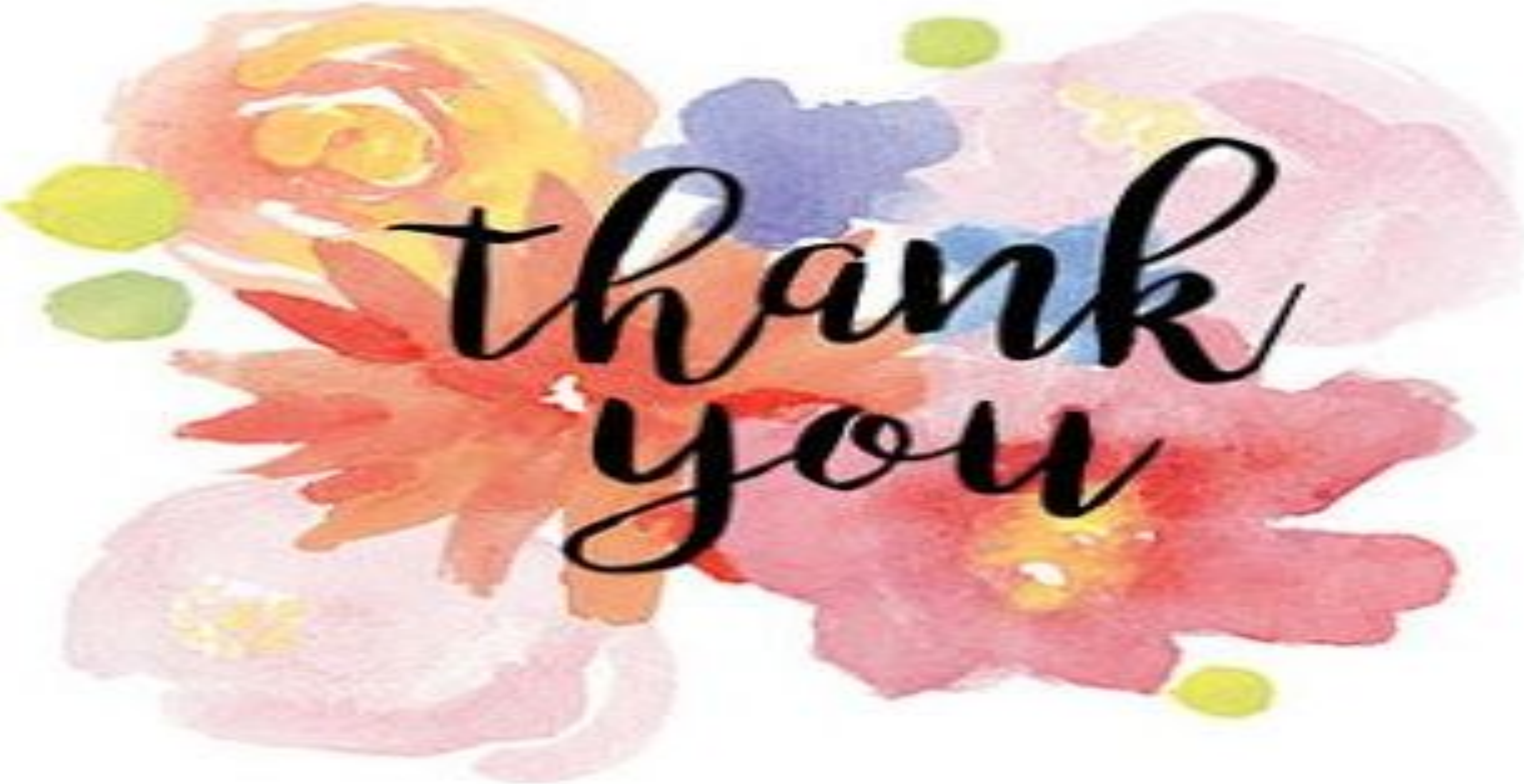
Ans : $y(0.6) = 0.6841$

Solve the following initial value problem using Runge-Kutta method of fourth order :

(i) $\frac{dy}{dx} = (1 + x)\, y$ with $y(0) = 1$ for $x = 0(0.2)\ 0.6$. $y(0.2) = 1.2247$, $y(0.4) = 1.5240$, $y(0.6) = 1.9581$

(ii) $y' = \frac{1}{x + y}$ with $y(0) = 1$ for $x = 0.5\ (0.5)\ 2$.

Ans : $y(0.5) = 1.3571$, $y(1.0) = 1.5873$, $y(1.5) = 1.7555$, $y(2.0) = 1.8956$


thank you