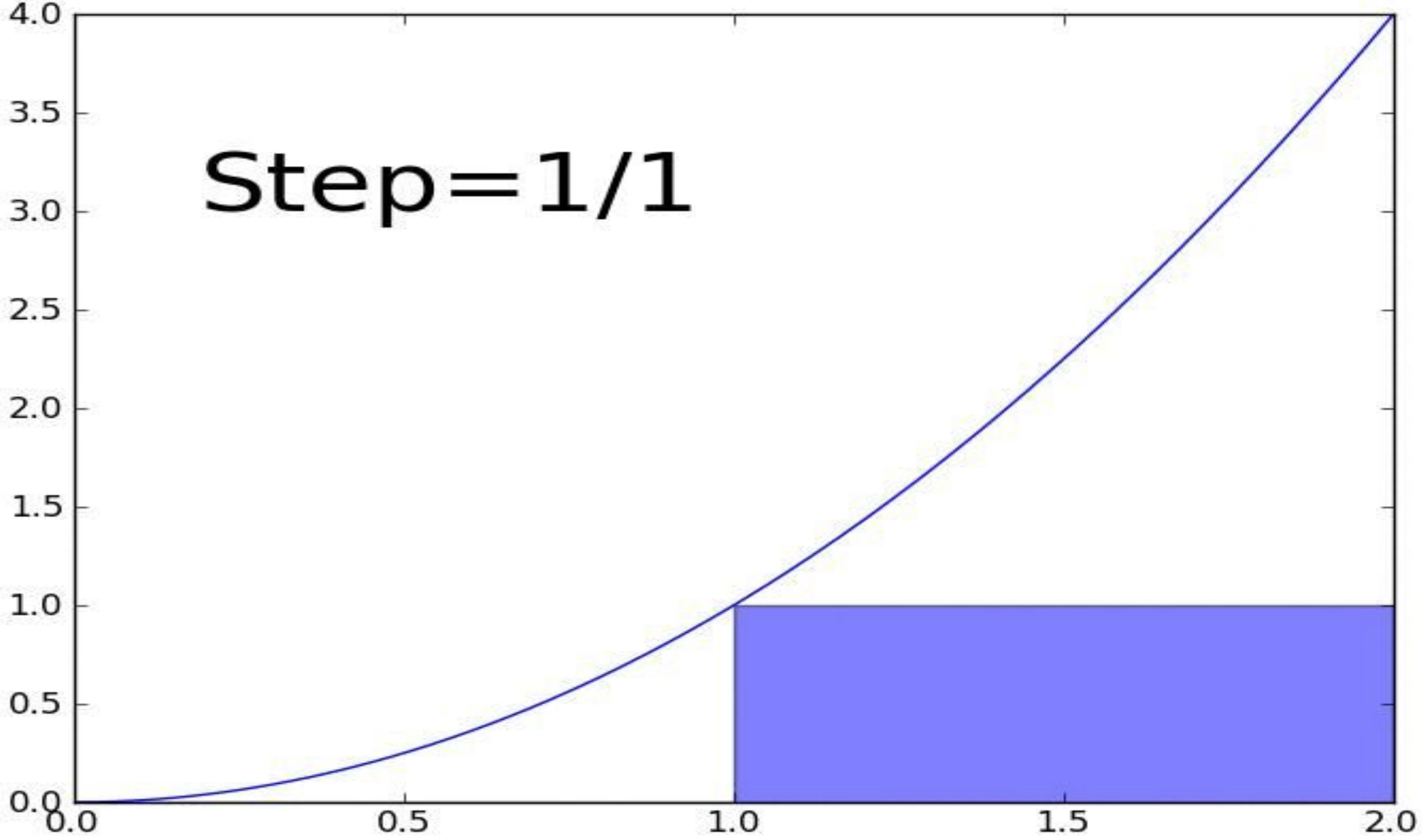
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4.0
3.5
3.0
2.5
2.0
1.5
1.0
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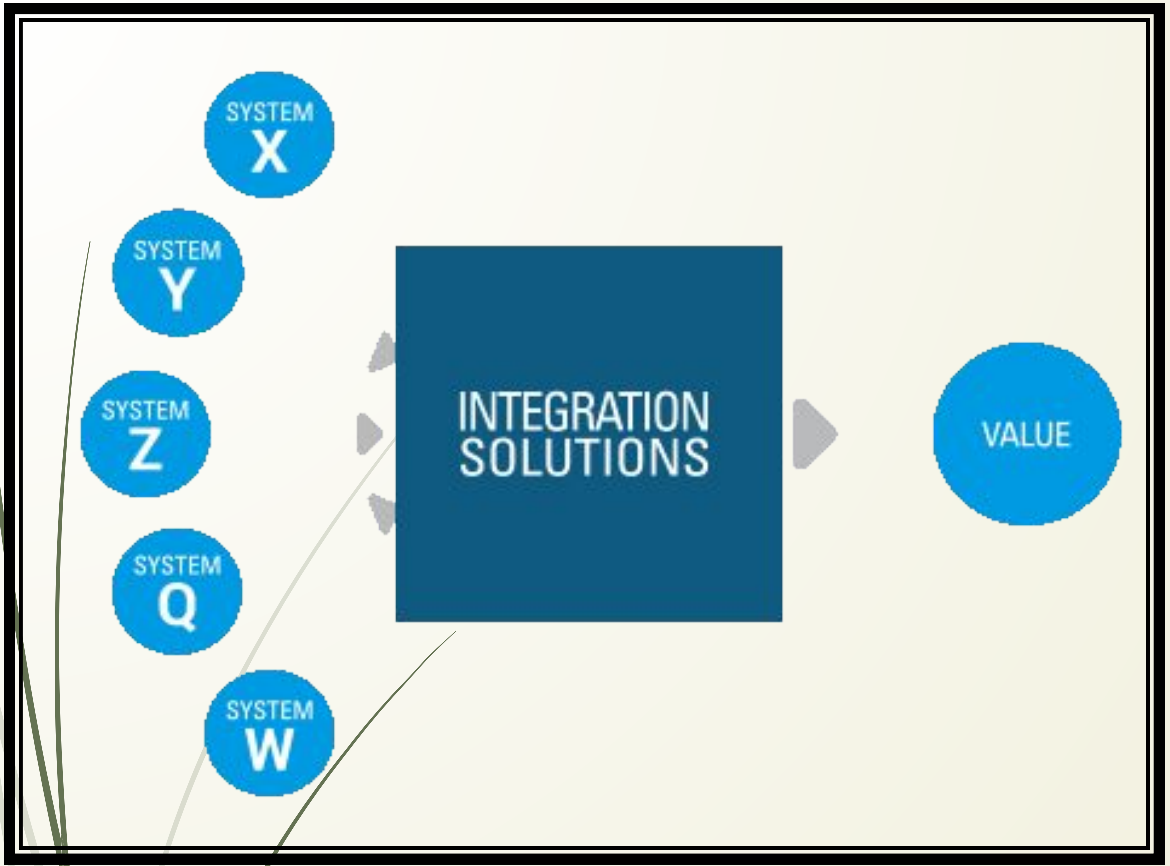
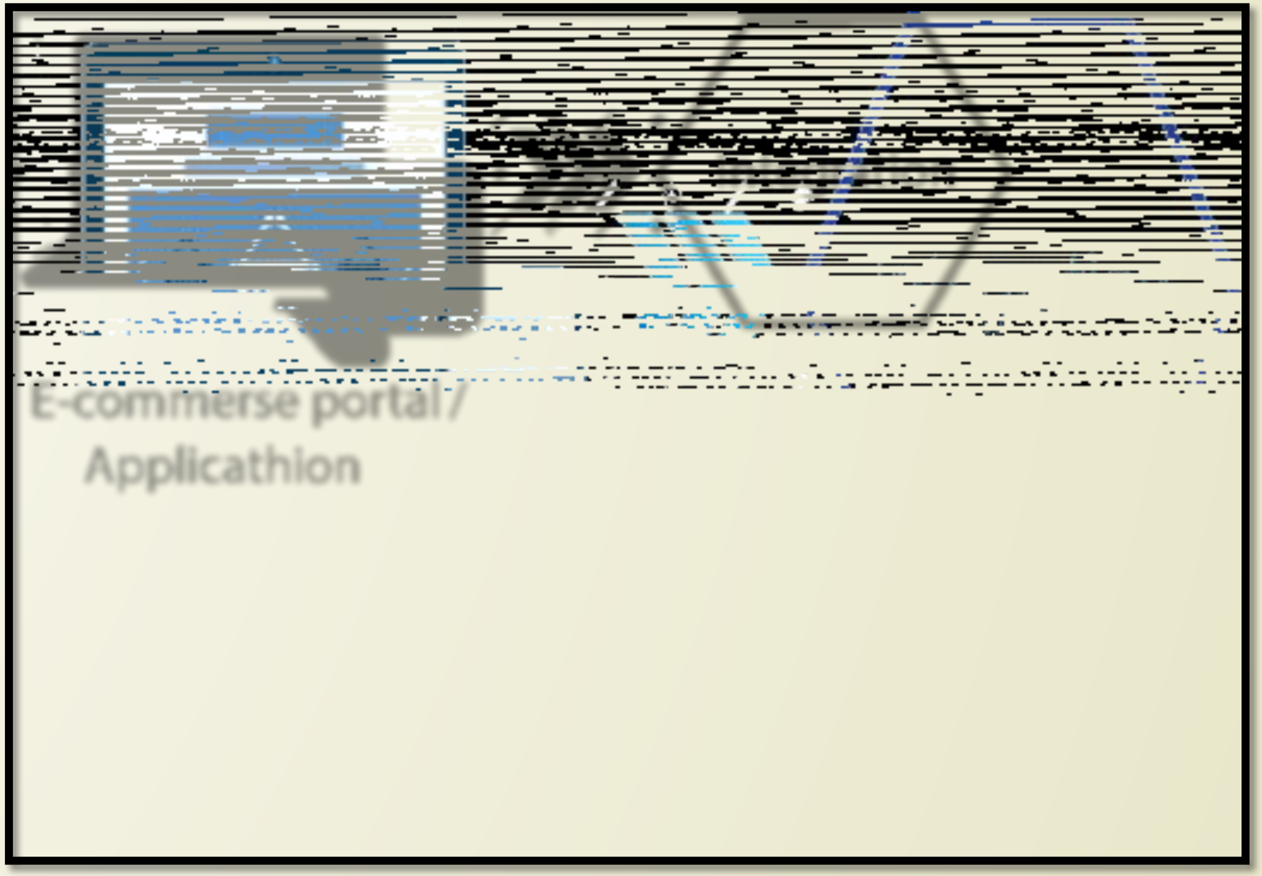
0.0
0.0
0.5
1.0
1.5
2.0





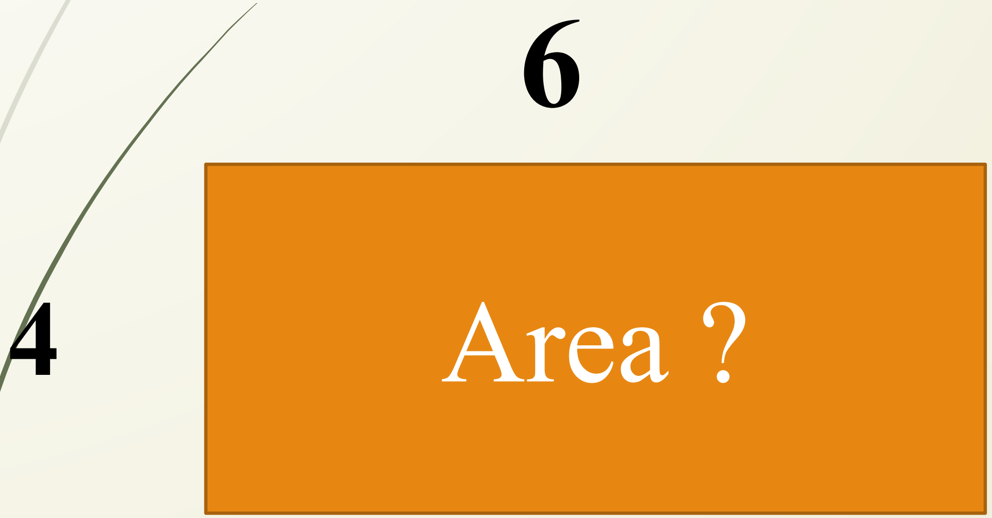
Numerical Integration

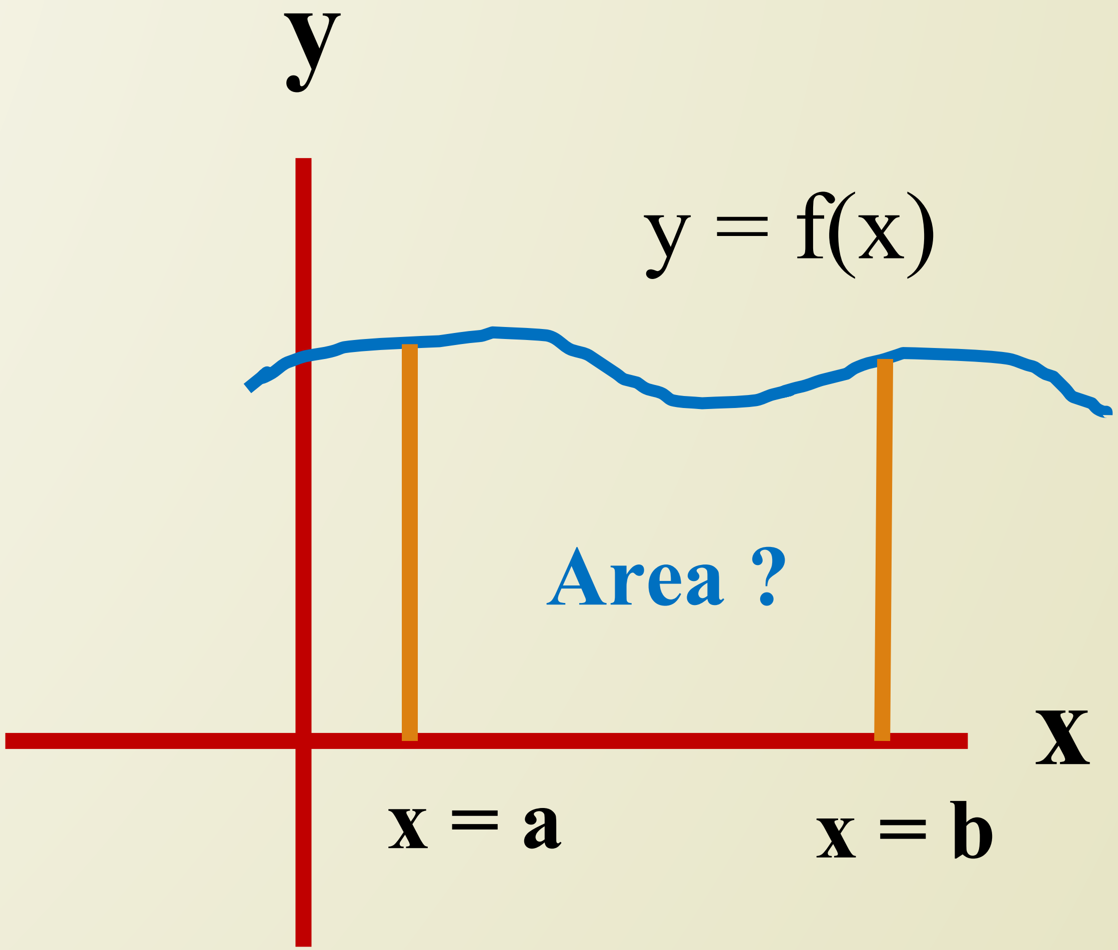
First part: General Formula for Numerical Integration (Trapezoidal)

Learning Results

- How to find out General Formula for the Numerical Integration ?
- How to get Trapezoidal Rules from General Formula ?
- How to solve the Trapezoidal problem ?

Basic Concept(Application of Integration)



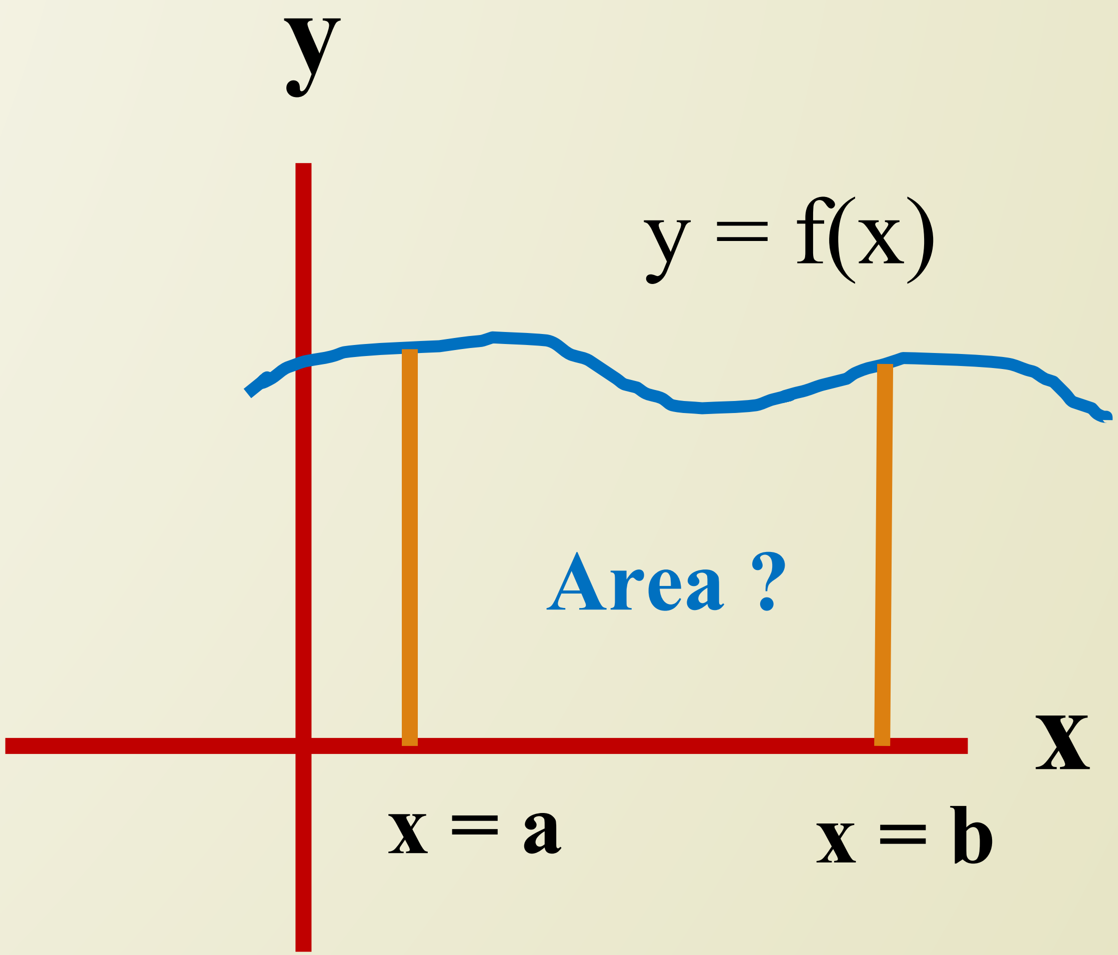


Basic Concept(Application of Integration)

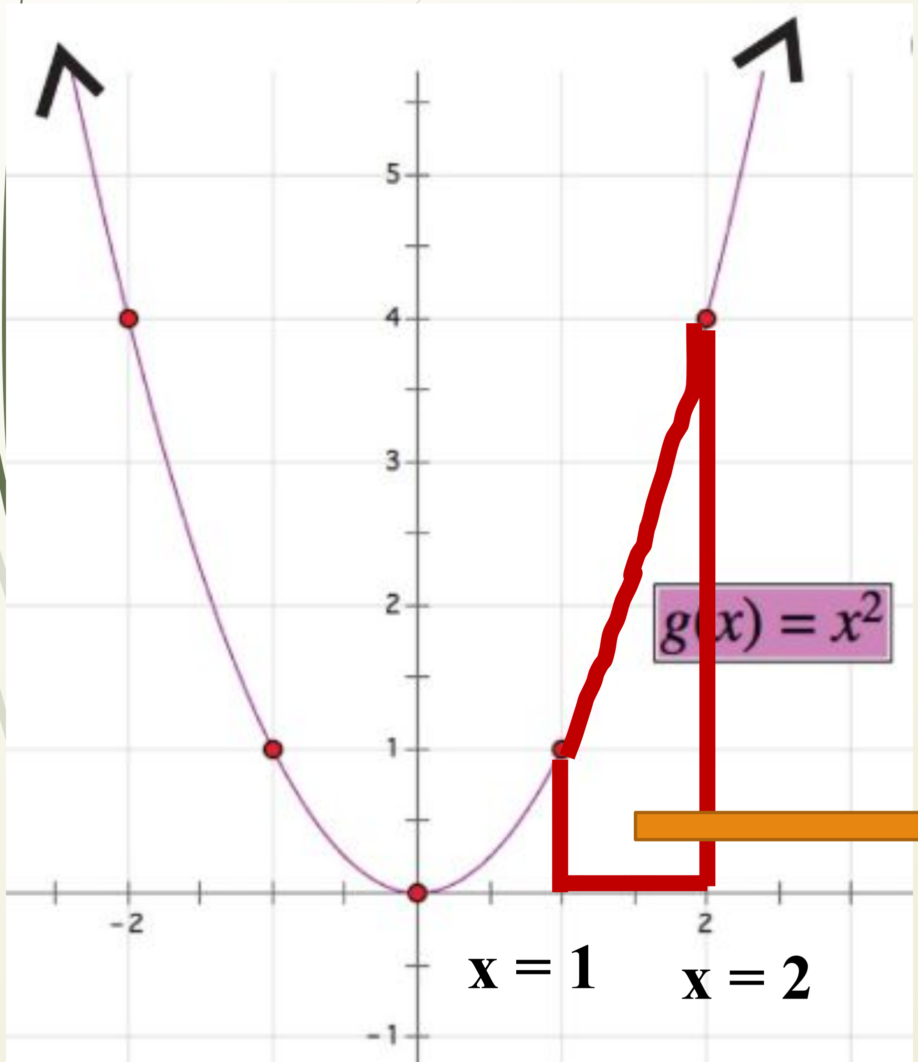
$$\int_{x=a}^{x=b} f$$

$$= [F(x)]_a^b$$

$$= F(a)\text{-}F(b)$$



$$y = x^2$$

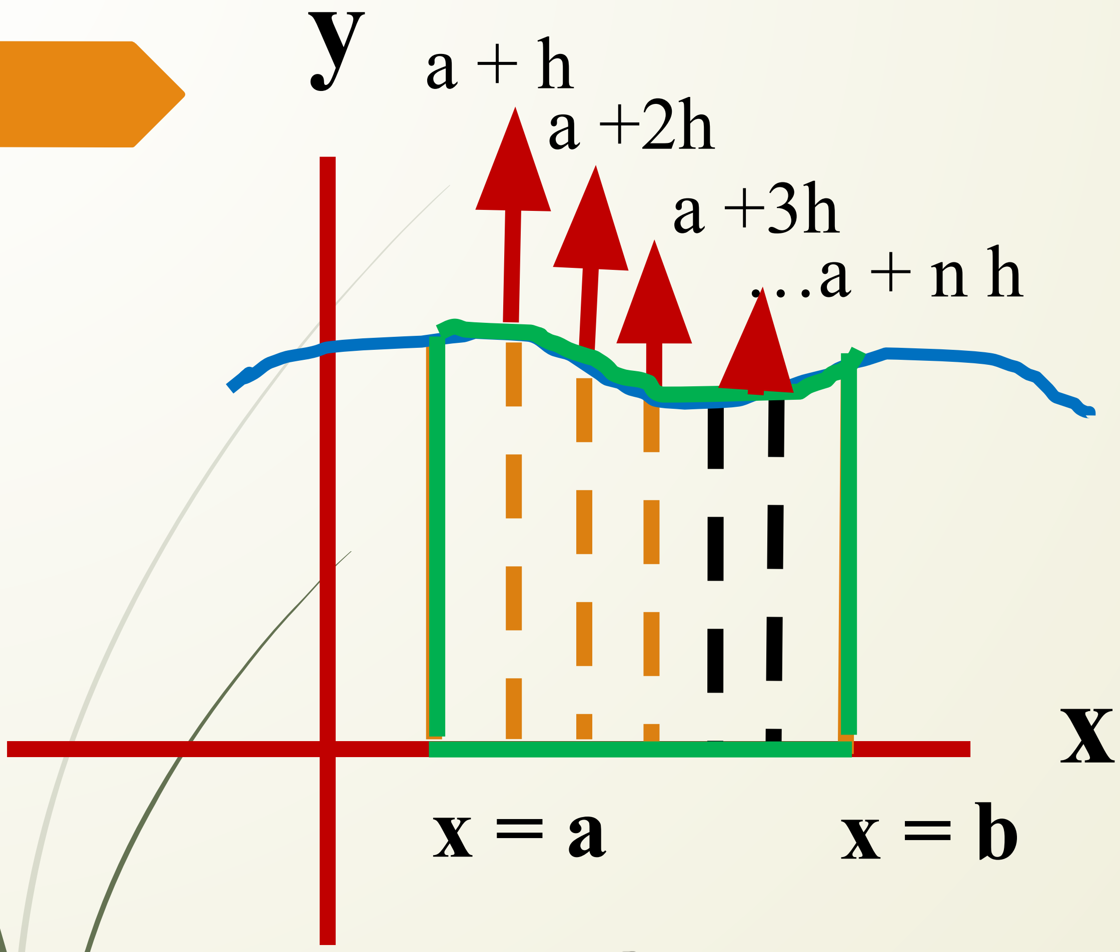


$$\text{Area} = \int_{x=a}^{x=b} f$$

$$= \int_1^2 x^2 dx = [\frac{x^3}{3}]_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1^3}{3} = \frac{7}{3}$$

Area ?



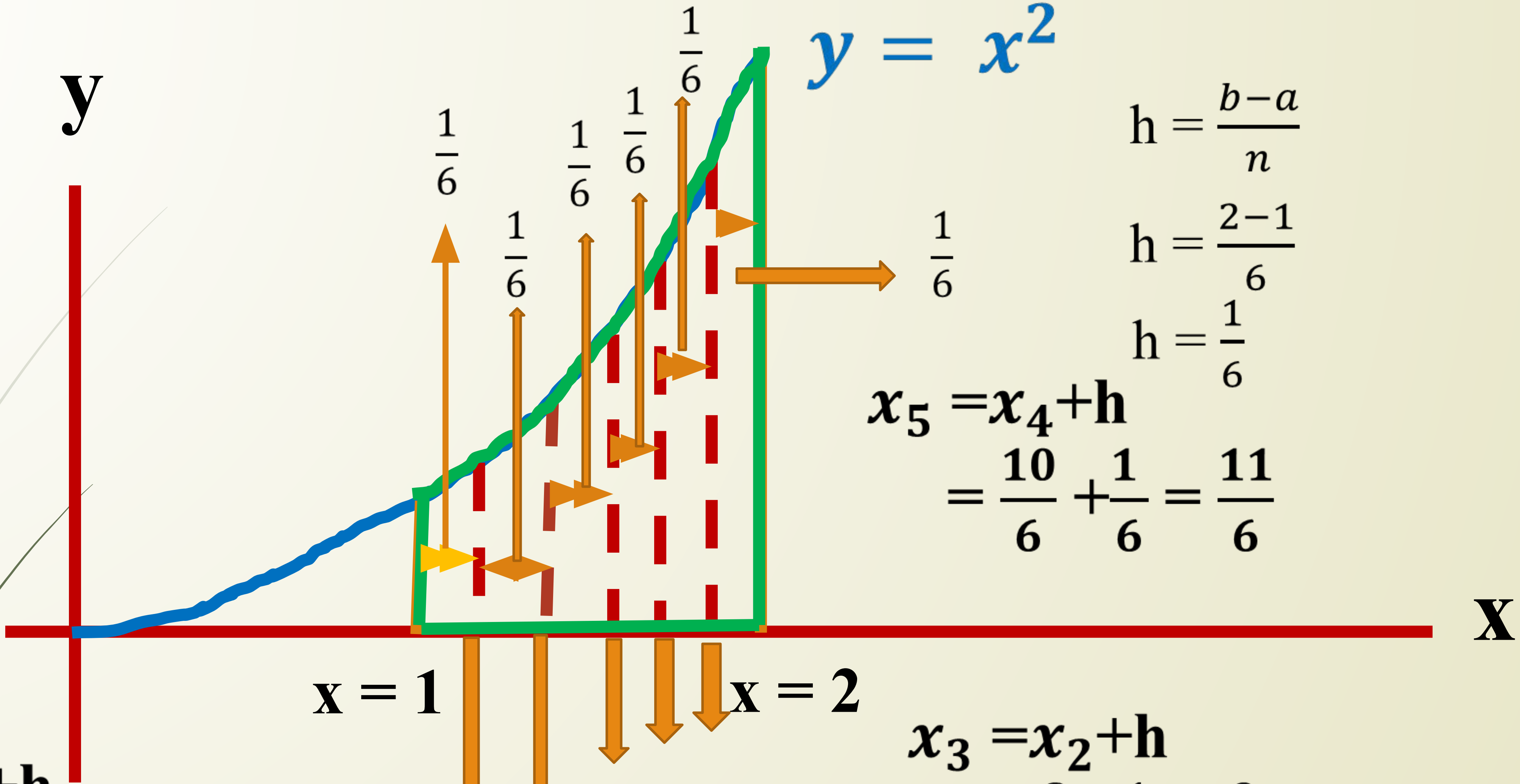
$$y = f(x) \qquad h = \frac{b-a}{n}$$

$$= \frac{upper\ limit - lower\ limit}{n}$$

Area $= \int_{x=a}^{x=b} f$

$$= \int_1^2 x^2 dx = [\frac{x^3}{3}]_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1^3}{3} = \frac{7}{3}$$



$$h = \frac{b-a}{n}$$

$$h = \frac{2-1}{6}$$

$$h = \frac{1}{6}$$

$$x_1 = 1 + h$$
$$= 1 + \frac{1}{6} = \frac{7}{6}$$

$$x_2 = x_1 + h$$
$$= \frac{7}{6} + \frac{1}{6} = \frac{8}{6}$$

$$x_3 = x_2 + h$$
$$= \frac{8}{6} + \frac{1}{6} = \frac{9}{6}$$

$$x_4 = x_3 + h$$
$$= \frac{9}{6} + \frac{1}{6} = \frac{10}{6}$$

$$x_5 = x_4 + h$$
$$= \frac{10}{6} + \frac{1}{6} = \frac{11}{6}$$

$$x_6 = x_5 + h$$
$$= \frac{11}{6} + \frac{1}{6} = \frac{12}{6} = 2$$

General formula for Numerical Integration

Let ,

$$I=\int_a^b y\,dx = \int_{x_0}^{x_0+nh} y\,dx \;\ldots\ldots.\,(1)$$

From Newton's Forward Interpolation Formula , We have,

General formula for Numerical Integration

$$y(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \ldots \ldots \ldots \ldots \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0$$

where, $u = \dfrac{x - x_0}{h}$

From (1) we get,

$$I = \int_{x_0}^{x_0+nh} \left[y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots \right] dx \quad \dots\dots\dots\dots\dots\dots\dots\dots(2)$$

Now we Know,

$$u = \frac{x - x_0}{h}$$

$$\Rightarrow x = x_0 + uh \qquad \therefore dx = h\,du$$

Limit Change:

When $x = x_0$ then $u = 0$

When $x = x_n$ then $u = n$

Therefore, above equation (2) takes the form,

$$I = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots + upto\,(n+1)terms \right] hdu$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots + upto\,(n+1)terms \right] du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{(u^2 - u)}{2!}\Delta^2 y_0 + \frac{(u^2 - u)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots + upto\,(n+1) terms \right] du$$

$$= h\int_0^n \left[y_0 + u\Delta y_0 + \frac{(u^2 - u)}{2!}\Delta^2 y_0 + \frac{(u^3 - 3u^2 + 2u)}{3!}\Delta^3 y_0 + \cdots\cdots\cdots\cdots + upto\,(n+1) terms \right] du$$

$$= h\left[y_0 u + \frac{u^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{u^4}{4} - u^3 + u^2\right)\Delta^3 y_0 + \cdots\cdots\cdots\cdots + upto\,(n+1) terms \right]_0^n$$

$$\therefore I = \int_a^b y dx = \int_{x_0}^{x_0+nh} y dx = h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots + upto\,(n+1) terms \right)$$

This Formula is known as general quadrature formula or General formula for numerical integration and also known as General Gauss -Legendre integration formula for equidistant ordinates.

Note:

1. This formula is used to compute $\int_a^b f(x)\,dx$

2. Putting $n = 1$ in above equation we obtain Trapezoidal rule

3. Putting $n = 2$ in above equation we obtain Simpson's $\frac{1}{3}$ Rule

4. Putting $n = 3$ in above equation we obtain Simpson's $\frac{3}{8}$ Rule

5. Putting $n = 4$ in above equation we obtain Boole's Rule

6. Putting $n = 6$ in above equation we obtain Weddle's Rule

Trapezoidal Rule

The general integration formula is

$$I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y\,dx$$

$$= h\left(ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\cdots\cdots + upto\,(n+1) terms \right)$$

Setting $n = 1$ in above equation and neglecting the second and higher order, we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

$$= h\left(y_0 + \frac{1}{2}(y_1 - y_0)\right)$$

$$= h\left(y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0\right) = h\left(\frac{1}{2}y_0 + \frac{1}{2}y_1\right)$$

Similarly, we can get,

$$\int_{x_0+h}^{x_0+2h} y\,dx = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_0+2h}^{x_0+3h} y\,dx = \frac{h}{2}(y_2 + y_3)$$

....

....

....

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \frac{h}{2}\left(y_{n-1} + y_n\right)$$

Adding these n integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left(y_0 + y_1\right) + \frac{h}{2}\left(y_1 + y_2\right) + \frac{h}{2}\left(y_2 + y_3\right) + \cdots\cdots\cdots\cdots + \frac{h}{2}\left(y_{n-1} + y_n\right)$$

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots\cdots\cdots\cdots + y_{n-1} + y_n\right)$$

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\cdots\cdots\cdots + y_{n-1})\right]$$

The above formula is known as the trapezoidal rule for numerical integration.

Shortly we can write,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2\sum_{k=1}^{n-1} y_k\right]$$

Problem

Evaluate $\int_0^6 f(x)\,dx$ by using trapezoidal rule where the values of $f(x)$ are given by the following table:

x	0	1	2	3	4	5	6
Y=f(x)	0.146	0.161	0.176	0.190	0.204	0.217	0.230

Solution:

Here upper limit is $b = 6$, lower limit is $a = 0$ and No. of subintervals $n = 6$.

Now,

$$h = \frac{6-0}{6} = 1$$

The values of the function y at each subinterval are given in the tabular form:

x	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	$x_4 = 4$	$x_5 = 5$	$x_6 = 6$
Y=f(x)	$y_0 = 0.146$	$y_1 = 0.161$	$y_2 = 0.176$	$y_3 = 0.190$	$y_4 = 0.204$	$y_5 = 0.217$	$y_6 = 0.230$

From trapezoidal rule we have

$$\int_a^b f(x)\,dx = \frac{1}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$\int_a^b f(x)\,dx = \frac{1}{2}[(0.146 + 0.230) + 2(0.161 + 0.176 + 0.190 + 0.204 + 0.217)]$$

$$\int_0^6 f(x)\,dx = 1.136$$

Practice Work

Calculate the value of the integral $I = \int_0^1 \frac{x\,dx}{1+x^2}$ by taking seven equidistant ordinates, using the trapezoidal rule. Find the exact value of I and then compare and comment on it.