Numerical Integration

2nd part : Simpson's one third

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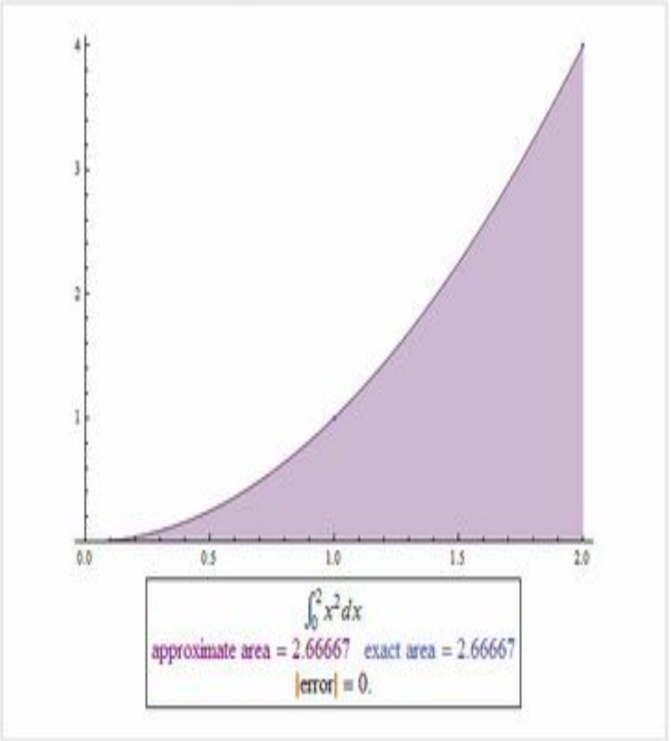

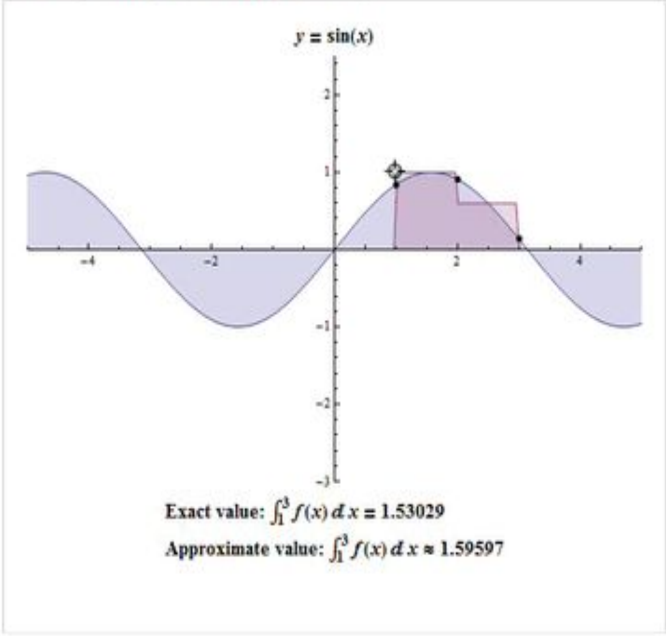

Simpson's $\frac{1}{3}$ Rule

The general Integration Formula ,

$$I = \int_{a}^{b} y \; dx = \int_{x_0}^{x_0+nh} y \, dx$$

Simpson's $\frac{1}{3}$ Rule

$$I = h\left[n\,y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \ldots\ldots\ldots + upto\ (n+1)\ terms\right]$$

Setting $n = 2$ in above equation and neglecting the third and higher order, we get

Simpson's $\frac{1}{3}$ Rule

$$\int_{x_0}^{x_0+2h} y\,dx = h\left(2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2!}\right)$$

$$= h\left(2y_0 + 2\Delta y_0 + \left(\frac{8}{3} - 2\right)\frac{\Delta^2 y_0}{2}\right)$$

$$= h\left(2y_0 + 2\Delta y_0 + \frac{\Delta^2 y_0}{3}\right)$$

Simpson's $\frac{1}{3}$ Rule

$$= h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}(\Delta y_1 - \Delta y_0)\right)$$

$$= h\left(2y_0 + 2(y_1 - y_0) + \frac{1}{3}\{(y_2 - y_1) - (y_1 - y_0)\}\right)$$

$$= h\left(2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right)$$

Simpson's $\frac{1}{3}$ Rule

$$= \frac{h}{3}\left(6y_0 + 6y_1 - 6y_0 + \left(y_2 - 2y_1 + y_0\right)\right)$$

$$\therefore \int_{x_0}^{x_0+2h} y\,dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Similarly, we can write,

Simpson's $\frac{1}{3}$ Rule

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

$$\int_{x_0+4h}^{x_0+6h} y\,dx = \frac{h}{3}\left(y_4 + 4y_5 + y_6\right)$$

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Simpson's $\frac{1}{3}$ Rule

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

Now adding the n integrals, we can write

Simpson's $\frac{1}{3}$ Rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$+ \frac{h}{3}(y_4 + 4y_5 + y_6) + \cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4$$

$$+ y_4 + 4y_5 + y_6 + \cdots\cdots\cdots\cdots\cdots\cdots + y_{n-2} + 4y_{n-1} + y_n]$$

Simpson's $\frac{1}{3}$ Rule

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\Big[(y_0 + y_n)$$

$$+ 4(y_1 + y_3 + y_5 + \cdots\cdots\cdots\cdots + y_{n-1})$$

$$+ 2(y_2 + y_4 + y_6 + \cdots\cdots\cdots\cdots + y_{n-2})\Big]$$

Simpson's $\frac{1}{3}$ Rule

The above formula is known as the Simpson's 1/3 rule for numerical integration.

Shortly we can write,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4\sum_{k=1,3,5,\ldots\ldots}^{n-1} y_k + 2\sum_{k=2,4,6,\ldots\ldots}^{n-2} y_k\right]$$

Note:

This formula is used only when the number of partitions of the interval of integration is even.

Mathematical Problems

Problem 01: Compute $\int_1^2 x^2 dx$ by Simpson's one third rule and compare with exact value.

Solution:

Given that the function is, $\int_1^2 x^2 dx$

Here upper limit is $b = 2$, lower limit is $a = 1$

and number of subintervals $n = 4$

Mathematical Problems

let $y = f(x) = x^2$

Now,

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

The values of the function y at each subinterval are given in the tabular form:

Mathematical Problems

x					

From the **Simpson's $\frac{1}{3}$ Rule , We have**

Mathematical Problems

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\big[(y_0 + y_n)$$

$$+ 4(y_1 + y_3 + y_5 + \cdots\cdots\cdots\cdots\cdots\cdots + y_{n-1})$$

$$+ 2(y_2 + y_4 + y_6 + \cdots\cdots\cdots\cdots\cdots\cdots + y_{n-2})\big]$$

Mathematical Problems

Now for $n = 4$ the above formula reduces to the following form,

$$\int_1^2 x^2 dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$= \frac{0.25}{3}\left[(1+4) + 4(1.5625 + 3.0625) + 2 \times 2.25\right]$$

$y_0 = 1$

$y_4 = 4$

$y_1 = 1.5625$

$y_2 = 2.25$

$y_3 = 3.0625$

Mathematical Problems

$$\therefore \int_1^2 x^2 \, dx = \frac{7}{3}$$

Now exact value is $\int_1^2 x^2 \, dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{1}{3}\left(2^3 - 1^3\right) = \frac{7}{3}$

It is shown that exact result and Simpson's $\frac{1}{3}$ Rule's result are exactly same so there is no error between two results.

Mathematical Problems

Problem 02:

Compute the definite integral $\int_{0.2}^{1.4} \left(\sin x - \ln x + e^x \right) dx$

by using various rules using 6 equidistant

sub-intervals correct up to three decimal places.

Mathematical Problems

Solution:

Given that the function is, $\int_{0.2}^{1.4} \left(\sin x - \ln x + e^x\right) dx$

Here upper limit is $b = 1.4$, lower limit is $a = 0.2$

No. of subintervals $n = 6$

Mathematical Problems

Let,

$$y = f(x) = \sin x - \ln x + e^x.$$

Now,

$$h = \frac{1.4 - 0.2}{6} = 0.2$$

The values of the function y at each subinterval are given in the tabular form:

Mathematical Problems

x	$x_0 = 0.2$	$x_1 = 0.4$	$x_2 = 0.6$	$x_3 = 0.8$	$x_4 = 1.0$	$x_5 = 1.2$	$x_6 = 1.4$
y	$y_0 = 3.0295$	$y_1 = 2.7975$	$y_2 = 2.8975$	$y_3 = 3.1660$	$y_4 = 3.5597$	$y_5 = 4.0698$	$y_6 = 4.7041$

From the **Simpson's $\frac{1}{3}$ Rule , We have**

Mathematical Problems

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\Big[(y_0 + y_n)$$

$$+4\left(y_1 + y_3 + y_5 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + y_{n-1}\right)$$

$$+2\left(y_2 + y_4 + y_6 + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + y_{n-2}\right)\Big]$$

Mathematical Problems

Now for $n = 6$ the above formula reduces to the following form,

$$\int_{0.2}^{1.4} \left(\sin x + \ln x - e^{x}\right) dx$$

$$= \frac{h}{3}\left[\left(y_0 + y_6\right) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{0.2}{3}\left[\left(3.0295 + 4.7041\right) + 4\left(2.7975 + 3.1660 + 4.0698\right) + 2\left(2.8975 + 3.5597\right)\right]$$

$$= \frac{0.2}{3}\left[7.7336 + 40.1332 + 12.9144\right]$$

Mathematical Problems

$$\therefore \int_{0.2}^{1.4} \left(\sin x + \ln x - e^{x} \right) dx \quad = 4.05208$$

Practice Work

1. Calculate the value of the integral $I = \int_0^1 \frac{x\,dx}{1+x^2}$ by taking seven equidistant ordinates, using the Simpson's 1/3 rule and trapezoidal rule. Find the exact value of I and then compare and comment on it.


Thank
You