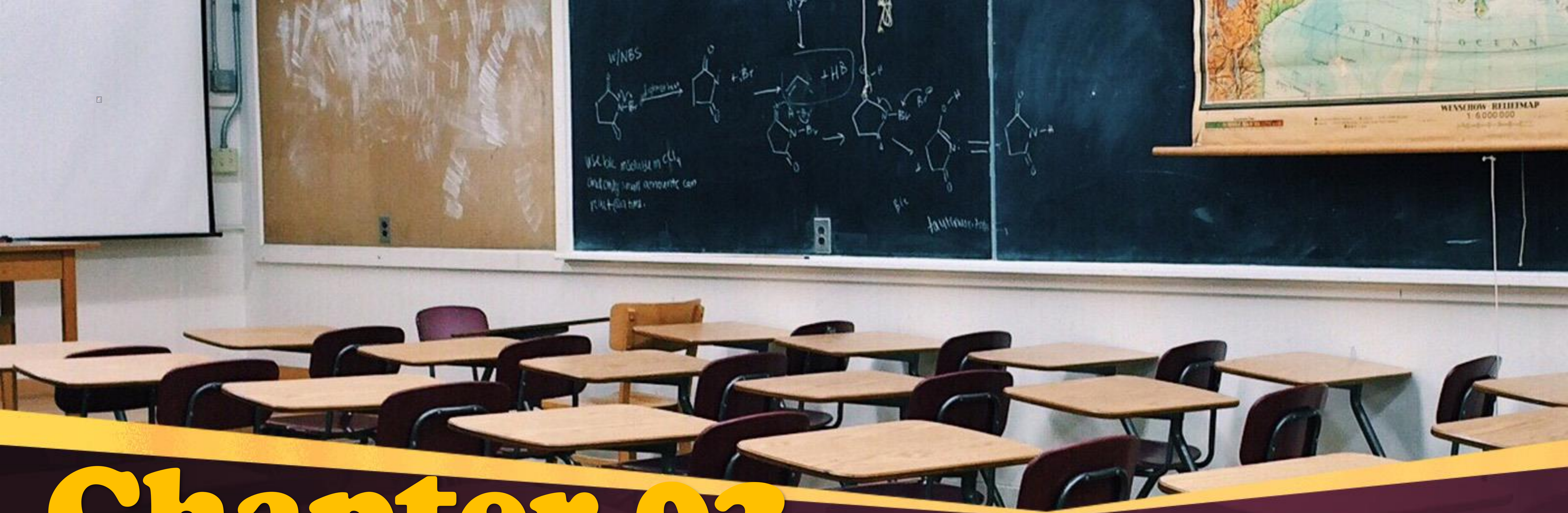


Chapter 02

Solution of Algebraic and Transcendental Equations

2. Newton - Raphson Method

Procedure of Newton – Raphson Method:

Step 1 : The given function = $f(x)$ and find $f'(x)$

$$Given, x^3 - 3x - 5 = 0$$

$$let, f(x) = x^3 - 3x - 5$$

Find $f(x)$ and $f'(x)$ in the step 1

$$f(x) = x^3 - 3x - 5$$

$$\therefore \frac{d}{dx} f(x) = \frac{d}{dx} x^3 - 3\frac{d}{dx} x - \frac{d}{dx} 5$$

$$\therefore f'(x) = 3x^2 - 3\times1 - 0$$

$$= 3x^2 - 3$$

Procedure of Newton – Raphson Method:

Find the Interval between (a , b) in the step 2

Step 2 : Choose , two real numbers a and b $such\,that,\ f(a)\times f(b)<0$

$$f(x)=x^3-3x-5$$

$$\therefore\ for\ x=a=2,\ f(a)=a^3-3\times a-5$$

$$\therefore\ f(2)=2^3-3\times 2-5$$

$$=8-11=-3<0$$

$$f(x)=x^3-3x-5$$

$$\therefore\ for\ x=b=3,\ f(b)=b^3-3\times b-5$$

$$\therefore\ f(b)=b^3-3\times b-5$$

$$=27-9-5=13>0$$

$$\therefore\ f(a)\times f(b)=f(2)\times f(3)=(-3)\times 13=-39<0$$

Since , $f(a) = f(2)$ is negative and $f(b) = f(3)$ is positive, So at least one real root lies between 2 and 3.

Step 3 : Find the follwing equation (1) by Newton – Raphson formula.

Newton – Raphson formula

we know that from Newton-Rapshon formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{.........(1)}$$

$$\because f(x) = x^3 - 3x - 5$$

$$\therefore f(x_n) = x^3_n - 3x_n - 5$$

$$\because x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\because f'(x) = 3x^2 - 3$$

$$\therefore f'(x_n) = 3x^2_n - 3$$

$$\therefore x_{n+1} = x_n - \frac{x^3_n - 3x_n - 5}{3x^2_n - 3} \quad \text{..........(2)}$$

Step 4 : *Let, the initial value*, $x = x_0 \in (a, b)$ *and putting the value*, $n=0$ in *the above equation* (1)

we are capable to find the successive improved approximations are as follows:

$$\because x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{........}(1)$$

putting $n=0$, *and*

let the initial value, $x = x_0$ *in equation* (1)

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Putting , $n = 0$ and then take initial value and then find first approximate value.

$$x_{n+1} = x_n - \frac{x^3{}_n - 3x_n - 5}{3x^2{}_n - 3} \text{......}(2)$$

For, $n = 0$ *and* $x = x_0 = 2$ (*say*) [From interval (2, 3)]

in equation (2)

$$x_{0+1} = x_0 - \frac{x^3{}_0 - 3x_0 - 5}{3x^2{}_0 - 3}$$

$$\Rightarrow x_1 = x_0 - \frac{x^3{}_0 - 3x_0 - 5}{3x^2{}_0 - 3}$$

$$\therefore x_1 = 2 - \frac{2^3 - 3\times 2 - 5}{3\times 2^2 - 3} = 2.3333$$

Step 4: Remain part

$$\because x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \text{........}(1)$$

putting $n=1$, *let the initial value*, $x=x_1$ *in equation* (1)

$$x_{1+1}=x_1-\frac{f(x_1)}{f'(x_1)}$$

$$\therefore x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

Similarly
find the others
approximate value.

$$x_{n+1}=x_n-\frac{x^3{}_n-3x_n-5}{3x^2{}_n-3} \text{.......}(2)$$

Again, putting $n=1$, $x=x_1$ *in equation* (2)

$$x_2=x_1-\frac{x^3{}_1-3x_1-5}{3x^2{}_1-3}$$

For $x=x_1=2.3333$

$$x_2=x_1-\frac{x^3{}_1-3x_1-5}{3x^2{}_1-3}$$

$$=2.3333-\frac{(2.3333)^3-3\times 2.3333-5}{3\times(2.3333)^2-3}=2.2806$$

Let see the above data by the following table :

No. of Iterations, *n*	$x_{n+1} = x_n - \frac{x^3_n - 3x_n - 5}{3x^2_n - 3}$	$x_{n+1} = x_n - \frac{x^3_n - 3x_n - 5}{3x^2_n - 3}$
$n=0$	$x_1 = x_0 - \frac{x^3_0 - 3x_0 - 5}{3x^2_0 - 3}$	$x_1 = 2 - \frac{2^3 - 3\times 2 - 5}{3\times 2^2 - 3} = 2.3333$
$n=1$	$x_2 = x_1 - \frac{x^3_1 - 3x_1 - 5}{3x^2_1 - 3}$	$x_2 = 2.3333 - \frac{(2.3333)^3 - 3\times 2.3333 - 5}{3\times (2.3333)^2 - 3} = 2.2806$

Step 4: Remain part

Similarly,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

… … ….

… … ….

… … ….

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

No. of Iterations, n	$x_{n+1} = x_n - \frac{x^3_n - 3x_n - 5}{3x^2_n - 3}$	$x_{n+1} = x_n - \frac{x^3_n - 3x_n - 5}{3x^2_n - 3}$
$n=2$	$x_3 = x_2 - \frac{x^3_2 - 3x_2 - 5}{3x^2_2 - 3}$	$x_3 = 2.2806 - \frac{(2.2806)^3 - 3\times(2.2806) - 5}{3\times(2.2806)^2 - 3} = 2.2790$
$n=3$	$x_4 = x_3 - \frac{x^3_3 - 3x_3 - 5}{3x^2_3 - 3}$	$x_4 = 2.2790 - \frac{(2.2790)^3 - 3\times 2.2790 - 5}{3\times(2.2790)^2 - 3} = 2.2790$

Step 5 : We shall continue this iterative process until the value of two successive approximation are approximately equal.

$$That\ is\ ,\ x_n \approx x_{n-1}\ \ or\ \ f(x_n) \approx 0.$$

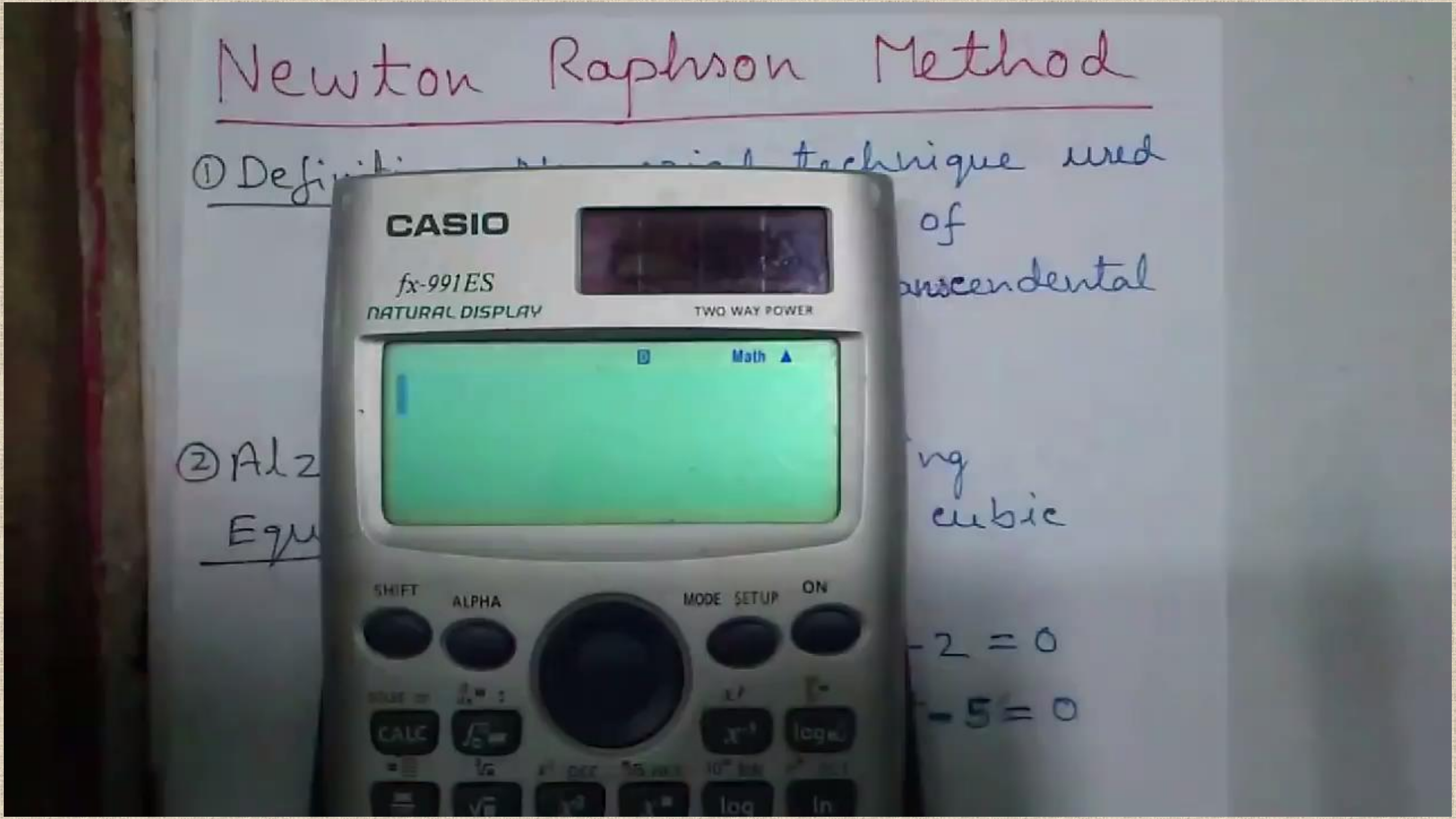
Since $x_n = x_{n-1}$ that is , $x_4 = x_3$ from the above table (slide no 9)

So the Newton - Rapshon method gives no new values of x

Therefore , the approximate root is correct to four decimal places.

Hence the require root is 2.2790 .

How to solve Newton –Raphson Method by using Calculator



Problem 1: Find the root of the equation $x\sin(x)+\cos(x)=0$, using Newton-Raphson method.

Solution : $Given\, that,\ x\sin(x)+\cos(x)=0$

Step 1

$$Let,\ f(x)=x\sin(x)+\cos(x)$$

$$\therefore f'(x)=\frac{d}{dx}x\sin(x)+\frac{d}{dx}\cos(x)$$

x = u

sin (x)= v

Step 1 :

$$f'(x)=\left[x\frac{d}{dx}\sin(x)+\sin(x)\frac{d}{dx}x\right]+\frac{d}{dx}\cos(x)$$

$$f'(x)=\left[x\cos(x)+\sin(x).1\right]-\sin(x)$$

$$\therefore\ f'(x)=x\cos(x)+\sin(x)-\sin(x)$$

$$=x\cos(x)$$

$$\frac{d}{dx}(uv)=u\frac{d}{dx}v+v\frac{d}{dx}u$$

$$\frac{d}{dx}\sin(x)=\cos(x)$$

$$\frac{d}{dx}x=1$$

$$\frac{d}{dx}\cos(x)=-\sin(x)$$

Hints: Calculator must be in radian Mode.

Step 2 :

$$\because f(x)=x\sin(x)+\cos(x)$$
$$for, x=2, then\ f(2)=2\sin(2)+\cos(2)$$
$$=1.4024>0$$

$$\because f(x)=x\sin(x)+\cos(x)$$
$$for, x=3, then\ f(3)=3\sin(3)+\cos(3)$$
$$=-0.5666<0$$

Since f (2) and f (3) are of opposite sign so at least one real root lies between 2 and 3.

Step 3 :

we know from Newton-Raphson method ,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1}=x_n-\frac{x_n\sin(x_n)+\cos(x_n)}{x_n\ \cos(x_n)}......(1)$$

$$\because f(x)=x\sin(x)+\cos(x)$$
$$\therefore f(x_n)=x_n\sin(x_n)+\cos(x_n)$$

$$\because f'(x)=x\cos(x)$$
$$\therefore f'(x_n)=x_n\cos(x_n)$$

Step 4 :

Now Putting $n = 0$ and let , the initial value , $x_0 = 2.5$ $in\,the\ above\,equation\,of$ (1)

Step 5 :

we are capable to find the successive improved approximations are as following table:

Hints: Calculator must be in radian Mode.

No. of Iterations, n	x_n	$x_{n+1} = x_n - \dfrac{x_n \sin(x_n) + \cos(x_n)}{x_n \cos(x_n)}$
$n = 0$	$x_0 = 2.5 (say)$	$x_1 = 2.5 - \dfrac{2.5 \sin(2.5) + \cos(2.5)}{2.5 \cos(2.5)} = 2.8470$
$n = 1$	$x_1 = 2.8470$	$x_2 = 2.8470 - \dfrac{2.8470 \sin(2.8470) + \cos(2.8470)}{2.8470 \cos(2.8470)} = 2.7992$
$n = 2$	$x_2 = 2.7992$	$x_3 = 2.7992 - \dfrac{2.7992 \sin(2.7992) + \cos(2.7992)}{2.7992 \cos(2.7992)} = 2.7984$
$n = 3$	$x_3 = 2.7984$	$x_4 = 2.7984 - \dfrac{2.7984 \sin(2.7984) + \cos(2.7984)}{2.7984 \cos(2.7984)} = 2.7984$

From the above table , we get ,

$$\therefore x_3 = 2.7984 \quad \text{and} \quad \therefore x_4 = 2.7984$$

$$\therefore x_4 \approx x_3$$

The approximate root is correct to four decimal places. Hence the require root is 2.7984.

You have to do mode on radian to your calculator

Problem 02: Find the real root of the equation $x^2 - 4\sin(x) = 0$ correct to four decimal places using Newton-Raphson method.

Solution :

$$Given\ that,\ x^2 - 4\sin(x) = 0$$

$$let,\ f(x) = x^2 - 4\sin(x)$$

$$\therefore\ f'(x) = \frac{d}{dx}x^2 - \frac{d}{dx}4\sin(x)$$

$$\therefore\ f'(x) = 2x - 4\cos(x)$$

$$\frac{d}{dx}x^n = n\,x^{n-1}$$

$$\therefore \frac{d}{dx}x^2 = 2\,x^{2-1} = 2x$$

$$\frac{d}{dx}a\,f(x) = a\frac{d}{dx}f(x) = a\,f'(x)$$

$$\therefore \frac{d}{dx}4\sin(x) = 4\frac{d}{dx}\sin(x) = 4\cos(x)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$\because f(x)=x^2-4\sin(x)$

$for\ x=a=1, then\ f(a)=a^2-4\sin(a)$

$\therefore f(1)=(1)^2-4\sin(1)=-2.3659<0$

$\because f(x)=x^2-4\sin(x)$

$for\ x=b=2, then\ f(b)=b^2-4\sin(b)$

$\therefore f(2)=(2)^2-4\sin(2)=0.3628>0$

Since $f(1)$ and $f(2)$ are of opposite sign , so at least one real root lies between 1 and 2.

We know from Newton-Raphson method ,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$\because f(x)=x^2-4\sin(x)$ $\qquad$ $\because f'(x)=2x-4\cos(x)$

$\therefore f(x_n)=x^2_n-4\sin(x_n)$ $\qquad$ $\therefore f'(x_n)=2x_n-4\cos(x_n)$

$$x_{n+1}=x_n-\frac{x^2_n-4\sin(x_n)}{2x_n-4\cos(x_n)}........(1)$$

Now Putting $n = 0$ and let the initial value , $x_0 = 1.5$ *in equation* (1)

we are capable to find the successive improved approximations are as following table:

No. of Iterations, n	x_n	$x_{n+1} = x_n - \frac{x^2{}_n - 4\sin(x_n)}{2x_n - 4\cos(x_n)}$
$n=0$	$x_0 = 1.5$	$x_1 = 1.5 - \frac{(1.5)^2 - 4\sin(1.5)}{2(1.5) - 4\cos(1.5)} = 2.1404$
$n=1$	$x_1 = 2.1404$	$x_2 = 2.1404 - \frac{(2.1404)^2 - 4\sin(2.1404)}{2(2.1404) - 4\cos(2.1404)} = 1.9520$
$n=2$	$x_2 = 1.9520$	$x_3 = 1.9520 - \frac{(1.9520)^2 - 4\sin(1.9520)}{2(1.9520) - 4\cos(1.9520)} = 1.9339$
$n=3$	$x_3 = 1.9339$	$x_4 = 1.9339 - \frac{(1.9339)^2 - 4\sin(1.9339)}{2(1.9339) - 4\cos(1.9339)} = 1.9338$
$n=4$	$x_4 = 1.9338$	$x_5 = 1.9338 - \frac{(1.9338)^2 - 4\sin(1.9338)}{2(1.9338) - 4\cos(1.9338)} = 1.9338$

From the above table , we get ,

$$x_4 = 1.9338 \quad \text{and} \quad x_5 = 1.9338$$

$$\therefore \; |x_5 - x_4| \approx 0$$

$$\therefore \; x_5 \approx x_4$$

The approximate root is correct to four decimal places.
Hence the require root is 1.9338.

Algorithm for Newton-Raphson method:

Steps	Task
01	Define $f(x)$
02	Define $f'(x)$
03	Read the initial value x_0
04	Set $n = 0$
05	Calculate $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
06	$If \quad \lvert x_n - x_{n-1} \rvert \approx 0.0001$ $Then\ go\ to\ step\ 8$ $elseif$ $n = n+1$ $Go\ to\ step\ 5$ $Endif$
07	Print x_n, the desired root
08	Stop

PRACTICE WORK

Find the root of the following equation by using Newton – Raphson method :

1. $2x = \ln x + 7$ *correct to* 4 *decimal places*

2. $3x + \sin x = e^x$ *correct to* 3 *decimal places.*

3. $\cos x = 3x - 1$ *correct to* 3 *decimal places.*

4. $e^{x} \tan x = 1$ correct to 3 decimal

5. $3x = \log_{10} x + 7$ *correct to* 4 *decimal places*