Separation of variables

If P and Q are function of two variables x and y, then the general first order and first degree ODE can be written as

$$P(x,y)+Q(x,y)\frac{dy}{dx}=0$$

$$\text{Or, } P(x,y)\,dx+Q(x,y)\,dy=0$$

If these equations can be written as

$$f_1(x)+f_2(y)\frac{dy}{dx}=0$$

$$\text{Or, } f_1(x)\,dx+f_2(y)\,dy=0$$

by some algebraic manipulation, then the variables are said to be separated. Integrated both sides,

$$\int f_1(x)\,dx+\int f_2(y)\,dy=c$$

$$\text{Or, } F_1(x)+F_2(y)=c.$$

Problem-1: Solve the ODE: $\frac{dy}{dx}=2xy$

Solution: Given that,

$$\frac{dy}{dx}=2xy$$

Separating variables we obtain,

$$\frac{dy}{y}=2xdx$$

Now, integrating,

$$\int\frac{dy}{y}=2\int xdx$$

$$or, \ln y=2\frac{x^2}{2}+\ln c$$

$$or, \ln y=x^2+\ln c$$

$$or, \ln y=\ln e^{x^2}+\ln c$$

$$or, \ln y=\ln\left(ce^{x^2}\right)$$

$$\therefore y=ce^{x^2}$$

Problem-2: Solve the ODE: $\frac{dy}{dx}=-\frac{xy}{x+1}$

Solution: Given that,

$$\frac{dy}{dx}=-\frac{xy}{x+1}$$

Separating variables we obtain,

$$\frac{dy}{y}=-\frac{xdx}{x+1}$$

$$or, \frac{dy}{y}=-\left(1-\frac{1}{x+1}\right)dx$$

Now, integrating,

$$\int\frac{dy}{y}=-\int\left(1-\frac{1}{x+1}\right)dx$$

$$or, \ln y = -x + \ln(x+1) + \ln c$$
$$or, \ln y = \ln e^{-x} + \ln(x+1) + \ln c$$
$$or, \ln y = \ln\left(ce^{-x}(x+1)\right)$$
$$\therefore y = ce^{-x}(x+1)$$

Problem-3: Solve the ODE: $\frac{dy}{dx} = \frac{x+1}{y^2}$

Solution: Given that,

$$\frac{dy}{dx} = \frac{x+1}{y^2}$$

Separating variables we obtain,

$$y^2 dy = (x+1)dx$$

Now, integrating,

$$\int y^2 dy = \int (x+1)dx$$
$$or, \frac{y^3}{3} = \frac{x^2}{2} + x + c$$

Problem-4: Solve the ODE: $\frac{dy}{dx} = x^2 y$

Solution: Given that,

$$\frac{dy}{dx} = x^2 y$$

Separating variables we obtain,

$$\frac{dy}{y} = x^2 dx$$

Now, integrating,

$$\int \frac{dy}{y} = \int x^2 dx$$
$$or, \ln y = \frac{x^3}{3} + \ln c$$
$$or, \ln y = \ln e^{\frac{x^3}{3}} + \ln c$$
$$or, \ln y = \ln\left(ce^{\frac{x^3}{3}}\right)$$
$$\therefore y = ce^{\frac{x^3}{3}}$$

Problem-5: Solve the ODE: $\frac{dy}{dx} = e^{x+y}$

Solution: Given that,

$$\frac{dy}{dx} = e^{x+y}$$
$$or, \frac{dy}{dx} = e^x . e^y$$

Separating variables we obtain,

$$\frac{dy}{e^y} = e^x dx$$
$$or, e^{-y} dy = e^x dx$$

Now, integrating,

$$\int e^{-y} dy = \int e^x dx$$
$$or, -e^{-y} = e^x + c$$

Problem-6: Solve the ODE: $\frac{dy}{dx} = e^{x-y}$

Solution: Given that,

$$\frac{dy}{dx} = e^{x-y}$$
$$or, \frac{dy}{dx} = e^x . e^{-y}$$

Separating variables we obtain,

$$\frac{dy}{e^{-y}} = e^x dx$$
$$or, e^y dy = e^x dx$$

Now, integrating,

$$\int e^y dy = \int e^x dx$$
$$or, e^y = e^x + c$$

Problem-7: Solve the ODE: $\frac{dy}{dx} = e^{2x+3y}$

Solution: Given that,

$$\frac{dy}{dx} = e^{2x+3y}$$
$$or, \frac{dy}{dx} = e^{2x} . e^{3y}$$

Separating variables we obtain,

$$\frac{dy}{e^{3y}} = e^{2x} dx$$
$$or, e^{-3y} dy = e^{2x} dx$$

Now, integrating,

$$\int e^{-3y} dy = \int e^{2x} dx$$
$$or, \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c$$

Problem-8: Solve the ODE: $x \frac{dy}{dx} = (1 - 2x^2) \tan y$

Solution: Given that,

$$x \frac{dy}{dx} = (1 - 2x^2) \tan y$$

Separating variables we obtain,

$$\frac{dy}{\tan y} = \frac{1 - 2x^2}{x} dx$$
$$or, \cot y \, dy = \left(\frac{1}{x} - 2x\right) dx$$

Now, integrating,

$$\int \cot y \, dy = \int \left(\frac{1}{x} - 2x\right) dx$$
$$or, \ln siny = \ln x - 2\frac{x^2}{2} + \ln c$$
$$or, \ln siny = \ln x - x^2 + \ln c$$
$$or, \ln siny = \ln x + \ln e^{-x^2} + \ln c$$
$$or, \ln \sin y = \ln\left(cx. e^{-x^2}\right)$$
$$\therefore \sin y = cx. e^{-x^2}$$

Problems for solution

1. $\cos y\, dy = sec^2 x\, dx$
2. $5cos^2 y\, dx + csc^2 x\, dy = 0$
3. $(e^y + 1)\cos x\, dy + e^y(\sin x + 1)dx = 0$
4. $x sin\, y\, dx = (x^2 + 1)\cos y\, dy$
5. $\frac{dy}{dx} = \frac{7xe^{2y}}{5+3x^2}$
6. $\frac{dy}{dx} = (1 + y^2)\sin 2x\, e^{5x}$

Some real life problem

(Q) The marginal cost function for producing x units is MC = 23+16x – $3x^2$ and the total cost for producing 1 unit is Rs.40. Find the total cost function and the average cost function.

Solution:

Let C(x) be the total cost function where x is the number of units of output. Then

$$\frac{dC}{dx} = MC = 23 + 16x - 3x^2$$

$$\therefore \int \frac{dC}{dx} dx = \int (23 + 16x - 3x^2) dx + k$$

$$C = 23x + 8x^2 - x^3 + k, \quad \text{where } k \text{ is a constant}$$

$$\text{At } x = 1,\ C(x) = 40 \quad \text{(given)}$$

$$23(1) + 8(1)^2 - 1^3 + k = 40 \Rightarrow k = 10$$

$$\therefore \quad \text{Total cost function } C(x) = 23x + 8x^2 - x^3 + 10$$

$$\text{Average cost function} = \frac{\text{Total cost function}}{x}$$

$$= \frac{23x + 8x^2 - x^3 + 10}{x}$$

$$\text{Average cost function} = 23 + 8x - x^2 + \frac{10}{x}$$

(Q) What is the general form of the demand equation which has a constant elasticity of -1?

Solution :

Let x be the quantity demanded at price p. Then the elasticity is given by

$$\eta_d = \frac{-p}{x}\frac{dx}{dp}$$

$$\text{Given } \frac{-p}{x}\frac{dx}{dp} = -1 \Rightarrow \frac{dx}{x} = \frac{dp}{p} \Rightarrow \int \frac{dx}{x} = \int \frac{dp}{p} + \log k$$

$$\Rightarrow \log x = \log p + \log k, \quad \text{where } k \text{ is a constant.}$$

$$\Rightarrow \log x = \log kp \Rightarrow x = kp \Rightarrow p = \frac{1}{k}x$$

$$\text{i.e. } p = cx, \quad \text{where } c = \frac{1}{k} \text{ is a constant}$$

(Q) The relationship between the cost of operating a warehouse and the number of units of items stored in it is given by $\frac{dy}{dx} = ax + b$ where C is the monthly cost of operating the warehouse and x is the number of units of items in storage. Find C as a function of x if C = C0 when x = 0.

Solution :

$$\text{Given } \frac{dC}{dx} = ax + b \quad \therefore \; dC = (ax + b)\, dx$$

$$\int dC = \int (ax + b)\, dx + k, \; (k \text{ is a constant})$$

$$\Rightarrow \quad C = \frac{ax^2}{2} + bx + k,$$

$$\text{when } x = 0, \quad C = C_0 \quad \therefore \; (1) \Rightarrow \quad C_0 = \frac{a}{2}(0) + b(0) + k$$

$$\Rightarrow k = C_0$$

Hence the cost function is given by

$$C = \frac{a}{2}x^2 + bx + C_0$$