Linear Differential Equation and integrating factor

If P and Q are only functions of x or constants then the differential equation of the form $\frac{dy}{dx} + Py = Q$ is called the first order linear differential equation.

INTEGRATING FACTOR (I.F)
A given differential equation may not be integrable as such. But it may become integrable when it is multiplied by a function. Such a function is called the **integrating factor (I.F).** Hence an integrating factor is one which changes a differential equation into one which is directly integrable.

Solve: $\boldsymbol{\frac{dy}{dx} + 2y = 4x}$

Solution: Given that,

$\frac{dy}{dx} + 2y = 4x$…………(i)

I.F=$e^{\int 2dx} = e^{2x}$

Multiplying equation (i) by the I.F. e^{2x} we get,

$e^{2x}\frac{dy}{dx} + 2ye^{2x} = 4xe^{2x}$

$or, \frac{d}{dx}(ye^{2x}) = 4xe^{2x}$

Now integrating,

$ye^{2x} = 4\int xe^{2x}dx$

$or, ye^{2x} = 4\left[x\frac{e^{2x}}{2} - \int 1\frac{e^{2x}}{2}dx\right]$

$or, ye^{2x} = 2xe^{2x} - 2\int e^{2x}dx$

$or, ye^{2x} = 2xe^{2x} - 2\frac{e^{2x}}{2} + c$

$or, ye^{2x} = 2xe^{2x} - e^{2x} + c$

$\therefore y = 2x - 1 + ce^{-2x}$

Solve: $\boldsymbol{\frac{dy}{dx} + y = e^{5x}}$

Solution: Given that,

$\frac{dy}{dx} + y = e^{5x}$………………(i)

I.F. =$e^{\int 1dx} = e^{x}$

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Multiplying equation (i) by the I.F. e^x we get,

$e^x \frac{dy}{dx} + ye^x = e^{5x}.e^x$

$or, \frac{d}{dx}(ye^x) = e^{6x}$

Now integrating,

$ye^x = \int e^{6x} dx$

$or, ye^x = \frac{e^{6x}}{6} + c$

$\therefore y = \frac{e^{5x}}{6} + ce^{-x}$

Solve: $\boldsymbol{\frac{dy}{dx} - y = e^{-5x}}$

Solution: Given that,

$\frac{dy}{dx} - y = e^{-5x}$………………(i)

I.F. $= e^{\int -1 dx} = e^{-x}$

Multiplying equation (i) by the I.F. e^{-x} we get,

$e^{-x} \frac{dy}{dx} + ye^{-x} = e^{-5x}.e^{-x}$

$or, \frac{d}{dx}(ye^{-x}) = e^{-6x}$

Now integrating,

$ye^{-x} = \int e^{-6x} dx$

$or, ye^{-x} = -\frac{e^{-6x}}{6} + c$

$\therefore y = -\frac{e^{-5x}}{6} + ce^x$

Solve: $\boldsymbol{\frac{dy}{dx} + y = \cos x}$

Solution: Given that,

$\frac{dy}{dx} + y = \cos x$……….(i)

I.F. $= e^{\int 1 dx} = e^x$

Multiplying equation (i) by the I.F. e^x we get,

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$e^x \frac{dy}{dx} + ye^x = e^x \cos x$

$or, \frac{d}{dx}(ye^x) = e^x \cos x$

Now integrating,

$ye^x = \int e^x \cos x dx$………………(ii)

Let $I = \int e^x \cos x dx$

$= e^x \sin x - \int e^x \sin x dx$

$= e^x \sin x - [e^x(-\cos x) - \int e^x(-\cos x) dx]$

$= e^x \sin x + e^x \cos x - \int e^x \cos x dx$

$= e^x(\sin x + \cos x) - I$

$or, 2I = e^x(\sin x + \cos x)$

$or, I = \frac{e^x}{2}(\sin x + \cos x)$

$\therefore \int e^x \cos x dx = \frac{e^x}{2}(\sin x + \cos x)$

From (ii) we get,

$ye^x = \frac{e^x}{2}(\sin x + \cos x) + c$

$\therefore y = \frac{1}{2}(\sin x + \cos x) + ce^{-x}$

Solve: $\boldsymbol{\frac{dy}{dx} + y = \sin x}$

Solution: Given that,

$\frac{dy}{dx} + y = \sin x$………………(i)

I.F. $= e^{\int 1 dx} = e^x$

Multiplying equation (i) by the I.F. e^x we get,

$e^x \frac{dy}{dx} + ye^x = e^x \sin x$

$or, \frac{d}{dx}(ye^x) = e^x \sin x$

Now integrating,

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$ye^x = \int e^x \sin x dx$…………..(ii)

Let $I = \int e^x \sin x dx$

$= e^x(-\cos x) - \int e^x(-\cos x)dx$

$= -e^x \cos x + \int e^x \cos x dx$

$= -e^x \cos x + e^x \sin x - \int e^x \sin x dx$

$= e^x(\sin x - \cos x) - I$

$or, 2I = e^x(\sin x - \cos x)$

$or, I = \frac{e^x}{2}(\sin x - \cos x)$

$\therefore \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x)$

From (ii) we get,

$ye^x = \frac{e^x}{2}(\sin x - \cos x) + c$

$\therefore y = \frac{1}{2}(\sin x - \cos x) + ce^{-x}$

Solve: $\boldsymbol{\frac{dy}{dx} + 2xy = 2xe^{x^2}}$

Solution: Given that,

$\frac{dy}{dx} + 2xy = 2xe^{x^2}$………………(i)

I.F. $= e^{\int 2x dx} = e^{2\frac{x^2}{2}} = e^{x^2}$

Multiplying equation (i) by the I.F. e^{x^2} we get,

$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 2xe^{x^2}.e^{x^2}$

$or, \frac{d}{dx}\left(ye^{x^2}\right) = 2xe^{2x^2}$

Now integrating,

$ye^{x^2} = \int 2xe^{2x^2}dx$

$= \int e^{2z}dz$

$= \frac{e^{2z}}{2} + c$

Let $x^2 = z$

$or, 2x = \frac{dz}{dx}$

$\therefore 2xdx = dz$

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$$= \frac{e^{2x^2}}{2} + c$$

$$\therefore y = \frac{e^{x^2}}{2} + ce^{-x^2}$$

Solve: $\boldsymbol{\frac{dy}{dx} = 2xy + x}$

Solution: Given that,

$$\frac{dy}{dx} = 2xy + x$$

$$or, \frac{dy}{dx} - 2xy = x \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

I.F. $= e^{\int -2x dx} = e^{-2\frac{x^2}{2}} = e^{-x^2}$

Multiplying equation (i) by the I.F. e^{-x^2} we get,

$$e^{-x^2}\frac{dy}{dx} - 2xye^{-x^2} = xe^{-x^2}$$

$$or, \frac{d}{dx}\left(ye^{-x^2}\right) = xe^{-x^2}$$

Now integrating,

$$ye^{-x^2} = \int xe^{-x^2} dx$$

$$= \int e^{-z}\frac{dz}{2}$$

$$= -\frac{e^{-z}}{2} + c$$

$$= -\frac{e^{-x^2}}{2} + c$$

Let $x^2 = z$

$$or, 2x = \frac{dz}{dx}$$

$$\therefore xdx = \frac{dz}{2}$$

$$\therefore y = -\frac{1}{2} + ce^{x^2}$$

Solve: $\boldsymbol{(1 + x^2)\frac{dy}{dx} - 2xy = (1 + x^2)^2}$

Solution: Given that,

$$(1 + x^2)\frac{dy}{dx} - 2xy = (1 + x^2)^2$$

$$or, \frac{dy}{dx} - \frac{2x}{1+x^2}y = 1 + x^2 \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

I.F. $= e^{\int -\frac{2x}{1+x^2}dx} = e^{-\ln(1+x^2)} = e^{\ln(1+x^2)^{-1}} = (1 + x^2)^{-1} = \frac{1}{1+x^2}$

Multiplying equation (i) by the I.F. $\frac{1}{1+x^2}$ we get,

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$\frac{1}{1+x^2}\frac{dy}{dx} - \frac{2x}{(1+x^2)^2}y = 1$

$or, \frac{d}{dx}\left(y\frac{1}{1+x^2}\right) = 1$

Now integrating,

$y\frac{1}{1+x^2} = \int 1dx$

$or, y\frac{1}{1+x^2} = x + c$

$\therefore y = (x+c)(1+x^2)$

Solve: $\boldsymbol{\frac{dy}{dx} + ycot\, x = \cot x}$

Solution: Given that,

$\frac{dy}{dx} + ycot\, x = \cot x$..............(i)

I.F. = $e^{\int \cot x dx} = e^{\ln \sin x} = \sin x$

Multiplying equation (i) by the I.F. $\sin x$ we get,

$sinx\frac{dy}{dx} + ycot\, x.\sin x = \cot x.\sin x$

$or, \frac{d}{dx}(ysin\, x) = \cot x.\sin x$

Now integrating,

$ysin\, x = \int \cot x.\sin x\, dx$

$or, ysin\, x = \int \frac{\cos x}{\sin x}\sin x\, dx$

$or, ysin\, x = \int \cos x dx$

$or, ysin\, x = \sin x + c$

$\therefore y = 1 + c.cosec\, x$

Solve: $\boldsymbol{\frac{dy}{dx} + ycot\, x = \sec x}$

Solution: Given that,

$\frac{dy}{dx} + ycot\, x = \sec x$.............(i)

I.F. =$e^{\int \cot x dx} = e^{\ln \sin x} = \sin x$

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Multiplying equation (i) by the I.F. $\sin x$ we get,

$sinx\frac{dy}{dx} + ycot\ x.\sin x = \sec x.\sin x$

$or, \frac{d}{dx}(ysin\ x) = \sec x.\sin x$

Now integrating,

$ysin\ x = \int \sec x.\sin x\ dx$

$or, ysin\ x = \int \frac{1}{\cos x}\sin x dx$

$or, ysin\ x = \int \tan x dx$

$or, ysin\ x = -\ln|\cos x| + c$

$\therefore\ y = [-\ln|\cos x| + c]cosec\ x$

Solve: $\boldsymbol{\frac{dy}{dx} + ycot\ x = \tan x}$

Solution: Given that,

$\frac{dy}{dx} + ycot\ x = \tan x$……………….(i)

I.F. = $e^{\int \cot x dx} = e^{\ln \sin x} = \sin x$

Multiplying equation (i) by the I.F. $\sin x$ we get,

$sinx\frac{dy}{dx} + ycot\ x.\sin x = \tan x.\sin x$

$or, \frac{d}{dx}(ysin\ x) = \tan x.\sin x$

Now integrating,

$ysin\ x = \int \tan x.\sin x\ dx$

$or, ysin\ x = \int \frac{\sin x}{\cos x}\sin x dx$

$or, ysin\ x = \int \frac{sin^2 x}{\cos x} dx$

$or, ysin\ x = \int \frac{1-cos^2 x}{\cos x} dx$

$or, ysinx = \int(\sec x - \cos x)dx$

$or, ysin\ x = \ln|\sec x + \tan x| - \sin x + c$

$\therefore\ y = \ln|\sec x + \tan x|.cosec\ x - 1 + c.cosec\ x$

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Problems for Solution

(i) $\frac{dy}{dx} - 3y = \cos x$

(ii) $\frac{dy}{dx} = -\frac{1}{x}y + x$

(iii) $\frac{dy}{dx} + \frac{1}{x}y = \cos x$

(iv) $\frac{dy}{dx} + \frac{2}{x}y = e^x$

(v) $\frac{dy}{dx} + y\cot x = \operatorname{cosec} x$

(vi) $\frac{dy}{dx} + y\sec x = \sin x$

(vii) $\frac{dy}{dx} + y\sec x = \cos x$

Real life problem

(Q) A computer manufacturing company of computer apparatus has found that the cost C of operating and maintaining the equipment is related to the length m of intervals between overhauls by the equation

$$m^2 \frac{dC}{dm} + 2mC = 2 \text{ and } C = 4 \text{ when } m = 2.$$

Find the relationship between C and m.

Solution :

Given $m^2 \frac{dC}{dm} + 2mC = 2$ or $\frac{dC}{dm} + \frac{2C}{m} = \frac{2}{m^2}$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{2}{m}$; $Q = \frac{2}{m^2}$

$\text{I.F} = e^{\int P dm} = e^{\int \frac{2}{m} dm} = e^{\log m^2} = m^2$

General solution is

$$C\,(\text{I.F}) = \int Q\,(\text{I.F})\, dm + k \quad \text{where } k \text{ is a constant}$$

$$Cm^2 = \int \frac{2}{m^2} m^2\, dm + k$$

$$Cm^2 = 2m + k$$

When $C = 4$ and $m = 2$, we have

$16 = 4 + k \quad \Rightarrow \quad k = 12$

∴ The relationship between C and m is

$Cm^2 = 2m + 12 = 2(m + 6)$

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