Bernoulli's Equation

If P and Q are only functions of x or constants, then the differential equation of the form $\frac{dy}{dx} + Py = Qy^n; n \neq 1$ is called Bernoulli's equation.

Solve: $\frac{dy}{dx} + 3x^2y = x^2y^3$

Solution: In this equation, $P(x) = 3x^2, Q(x) = x^2, n = 3$

Let us transform the above equation by the following transformation

$$z = y^{1-n} = y^{1-3} = y^{-2} = \frac{1}{y^2}$$

After the transformation the transformed equation stands,

$$\frac{dz}{dx} + (1-3)3x^2z = (1-3)x^2$$

$$or, \frac{dz}{dx} - 6x^2z = -2x^2, \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

which is a linear differential equation in z.

Now, I.F.=$e^{\int -6x^2dx} = e^{-6\frac{x^3}{3}} = e^{-2x^3}$

Now, multiplying the equation (i) by the I.F. e^{-2x^3},

$$e^{-2x^3}\frac{dz}{dx} - 6x^2ze^{-2x^3} = -2x^2e^{-2x^3}$$

$$or, \frac{d}{dx}\left(ze^{-2x^3}\right) = -2x^2e^{-2x^3}$$

Now integrating,

$ze^{-2x^3} = -2\int x^2e^{-2x^3}dx$

$= -2\int e^{-2t}\frac{dt}{3}$

$= -\frac{2}{3}\int e^{-2t}dt$

$= -\frac{2}{3}\left(\frac{e^{-2t}}{-2}\right) + c$

$ze^{-2x^3} = \frac{1}{3}e^{-2x^3} + c$

Let $x^3 = t$

$or, 3x^2 = \frac{dt}{dx}$

$or, x^2dx = \frac{dt}{3}$

$or, z = \frac{1}{3} + ce^{2x^3}$

$\therefore \frac{1}{y^2} = \frac{1}{3} + ce^{2x^3}$

Solve: $\boldsymbol{\frac{dy}{dx} + 2xy = xy^2}$

Solution: In this equation, $P(x) = 2x, Q(x) = x, n = 2$

Let us transform the above equation by the following transformation

$$z = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}$$

After the transformation the transformed equation stands,

$$\frac{dz}{dx} + (1-2)2xz = (1-2)x$$

$$or, \frac{dz}{dx} - 2xz = -x, \text{ …………………(i)}$$

which is a linear differential equation in z.

Now, I.F. $= e^{\int -2x dx} = e^{-2\frac{x^2}{2}} = e^{-x^2}$

Now, multiplying the equation (i) by the I.F. e^{-x^2},

$$e^{-x^2}\frac{dz}{dx} - 2xze^{-x^2} = -xe^{-x^2}$$

$$or, \frac{d}{dx}\left(ze^{-x^2}\right) = -xe^{-x^2}$$

Now integrating,

$ze^{-x^2} = -\int xe^{-x^2}dx$

$= -\int e^{-t}\frac{dt}{2}$

$= -\frac{1}{2}\int e^{-t}dt$

$= \frac{1}{2}e^{-t} + c$

Let $x^2 = t$

$or, 2x = \frac{dt}{dx}$ $or, xdx = \frac{dt}{2}$

$ze^{-x^2} = \frac{1}{2}e^{-x^2} + c$

$or, z = \frac{1}{2} + ce^{x^2}$

$\therefore \frac{1}{y} = \frac{1}{2} + ce^{x^2}$

Solve: $\boldsymbol{\frac{dy}{dx} + \frac{1}{x}y = xy^2}$

Solution: In this equation, $P(x) = \frac{1}{x}, Q(x) = x, n = 2$

Let us transform the above equation by the following transformation

$$z = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}$$

After the transformation the transformed equation stands,

$$\frac{dz}{dx} + (1-2)\frac{1}{x}z = (1-2)x$$

$$or, \frac{dz}{dx} - \frac{1}{x}z = -x, \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

which is a linear differential equation in z.

Now, I.F. $= e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

Now, multiplying the equation (i) by the I.F. $\frac{1}{x}$,

$$\frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2}z = -1$$

$$or, \frac{d}{dx}\left(z\frac{1}{x}\right) = -1$$

Now integrating,

$$z\frac{1}{x} = -\int dx$$

$$or, z\frac{1}{x} = -x + c$$

$$or, z = -x^2 + cx$$

$$\therefore \frac{1}{y} = -x^2 + cx$$

Solve: $\boldsymbol{\frac{dy}{dx} + \frac{1}{x}y = x\sqrt{y}}$

Solution: Given that,

$$\frac{dy}{dx} + \frac{1}{x}y = x\sqrt{y}$$

$$or, \frac{dy}{dx} + \frac{1}{x}y = xy^{\frac{1}{2}}$$

In this equation, $P(x) = \frac{1}{x}, Q(x) = x, n = \frac{1}{2}$

Let us transform the above equation by the following transformation

$$z = y^{1-n} = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$$

After the transformation the transformed equation stands,

$$\frac{dz}{dx} + \left(1 - \frac{1}{2}\right)\frac{1}{x}z = \left(1 - \frac{1}{2}\right)x$$

$$or, \frac{dz}{dx} + \frac{1}{2x}z = \frac{1}{2}x, \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

which is a linear differential equation in z.

Now, I.F. $= e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x} = x^{\frac{1}{2}}$

Now, multiplying the equation (i) by the I.F. $x^{\frac{1}{2}}$,

$$x^{\frac{1}{2}}\frac{dz}{dx} + \frac{1}{2x}zx^{\frac{1}{2}} = \frac{1}{2}x.x^{\frac{1}{2}}$$

$$or, \frac{d}{dx}\left(zx^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{3}{2}}$$

Now integrating,

$$zx^{\frac{1}{2}} = \int \frac{1}{2}x^{\frac{3}{2}}\,dx$$

$$= \frac{1}{2}\int x^{\frac{3}{2}}dx$$

$$= \frac{1}{2}\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$zx^{\frac{1}{2}} = \frac{1}{5}x^{\frac{5}{2}} + c$$

$$or, z = \frac{1}{5}x^2 + cx^{-\frac{1}{2}}$$

$$\therefore y^{\frac{1}{2}} = \frac{1}{5}x^2 + cx^{-\frac{1}{2}}$$

Problems for Solution

(i) $\frac{dy}{dx} - y = x^3\sqrt[3]{y}$

(ii) $y\frac{dy}{dx} - y^2 = e^x$