Application of first order ODE

Problem-01: The population of bacteria in a culture grows at a rate that is proportional to the number at present. Initially, it has p_0 number of bacteria, and after 1 hr the number of bacteria is measured to be $\frac{3}{2}p_0$.

(a). What is the number of bacteria after t hr?

(b). Determine the necessary time for number of bacteria to triple.

Solution: Let, $p(t)$ be the number of bacteria at any time *t* hr.

Since, the rate of growth of bacteria is proportional to the number of bacteria $p(t)$,

So $\quad \frac{dp}{dt} \propto p$

$$or,\ \frac{dp}{dt} = kp \ \ldots\ldots\ldots(1)$$

where, k is a proportional constant.

According to the question we have,

$$p(0) = p_0 \ \ldots\ldots\ldots (2)$$

$$p(1) = \frac{3}{2}p_0 \ \ldots\ldots\ldots (3)$$

Now from Eq. (1) we can write,

$$\frac{dp}{p} = kdt$$

$$or,\ \int\frac{dp}{p} = k\int dt$$

$$or,\ \ln p = kt + \ln c$$

$$or,\ \ln p = \ln e^{kt} + \ln c$$

$$or,\ \ln p = \ln ce^{kt}$$

$$\therefore\ p = ce^{kt} \ \ldots\ldots\ldots(4)$$

From Eq. (2) & Eq.(4) we have,

$$p_0 = ce^{k.0}$$

$$\therefore\ c = p_0$$

From Eq. (3) & Eq.(4) we have,

$$\frac{3}{2}p_0 = p_0 e^k$$

$$or,\ e^k = \frac{3}{2}$$

$$or,\ k = \ln\left(\frac{3}{2}\right)$$

$$\therefore\ k = 0.4055$$

Using the values of c & k in Eq.(4) we have,

$$p = p_0 e^{0.4055t}\ \ldots\ldots\ldots(5)$$

This is the number of bacteria after t hr.

Again, let after $t = t_1$ hr the number of bacteria will be triple. i.e, $p(t_1) = 3p_0$.

Now from Eq. (4) we have,

$$3p_0 = p_0 e^{0.4055t_1}$$

$$or,\ e^{0.4055t_1} = 3$$

$$or,\ 0.4055t_1 = \ln 3$$

$$or,\ t_1 = \frac{\ln 3}{0.4055}$$

$$or,\ t_1 = 2.71\ hr$$

This is the required time. (**ans.**)

Problem-02: The population of bacteria in a culture grows at a rate that is proportional to the number at present. Initially, there are 600 bacteria, and after 3 hr there are 10,000 bacteria.

(a). What is the number of bacteria after t hr?

(b). What is the number of bacteria after 5 hr?

(c). When will the number of bacteria reach 24,000?

Solution: Let, $p(t)$ be the number of bacteria at any time t hr.

Since, the rate of growth of bacteria is proportional to the number of bacteria $p(t)$,

So $$\frac{dp}{dt} \propto p$$

$$or, \ \frac{dp}{dt} = kp \ \ldots\ldots\ldots(1)$$

where, k is a proportional constant.

According to the question we have,

$$p(0) = 600 \ \ldots\ldots\ldots(2)$$

$$p(3) = 10,000 \ \ldots\ldots\ldots(3)$$

Now from Eq. (1) we can write,

$$\frac{dp}{p} = kdt$$

$$or, \int \frac{dp}{p} = k\int dt$$

$$or, \ \ln p = kt + \ln c$$

$$or, \ \ln p = \ln e^{kt} + \ln c$$

$$or, \ \ln p = \ln ce^{kt}$$

$$\therefore \ p = ce^{kt} \ \ldots\ldots\ldots(4)$$

From Eq. (2) & Eq.(4) we have,

$$600 = ce^{k.0}$$

$$\therefore \ c = 600$$

From Eq. (3) & Eq.(4) we have,

$$10,000 = 600e^{3k}$$

$$or,\ e^{3k} = \frac{10,000}{600}$$

$$or,\ e^{3k} = 16.667$$

$$or,\ 3k = \ln(16.667)$$

$$or,\ k = \frac{1}{3}\ln(16.667)$$

$$\therefore\ k = 0.9378$$

Using the values of *c* & *k* in Eq.(4) we have,

$$p = 600e^{0.9378t} \ \ldots\ldots\ldots(5)$$

This is the number of bacteria after *t* hr.

Again, For $t = 5$ we get from Eq. (5),

$$p = 600e^{0.9378\times 5}$$

$$\therefore\ p = 65246.63$$

This is the number of bacteria after 5 hr.

Again, let after $t = t_1$ hr the number of bacteria will be 24,000. i.e, $p(t_1) = 24,000$.

Now from Eq. (4) we have,

$$24,000 = 600e^{0.9378t_1}$$

$$or,\ e^{0.9378t_1} = 40$$

$$or,\ 0.9378t_1 = \ln(40)$$

$$or,\ t_1 = \frac{\ln(40)}{0.9378}$$

$$or,\ t_1 = 3.93\ hr$$

This is the required time. (**ans.**)

Problem-03: Radioactive substances decay at a rate that is proportional to the amount present. The half-life of a substance is the time required for a given amount to be reduced by one-half. The half-life of cesium-137 is 30 years. Suppose we have 100 mg sample.

(a). Find the mass that remains after *t* years.

(b). How much of the sample remains after 100 years?

(c). After how long will only 1 mg remain?

Solution: Let, $p(t)$ be the amount of cesium-137 present in the sample at any time *t* hr.

Since, the rate of decay of cesium-137 is proportional to the amount $p(t)$,

So $$\frac{dp}{dt} \propto p$$

$$or, \frac{dp}{dt} = kp \quad \dots\dots\dots(1)$$

where, k is a proportional constant.

According to the question we have,

$$p(0) = 100 \quad \dots\dots\dots (2)$$

$$p(30) = 50 \quad \dots\dots\dots (3)$$

Now from Eq. (1) we can write,

$$\frac{dp}{p} = kdt$$

$$or, \int \frac{dp}{p} = k\int dt$$

$$or, \ln p = kt + \ln c$$

$$or, \ln p = \ln e^{kt} + \ln c$$

$$or, \ln p = \ln ce^{kt}$$

$$\therefore p = ce^{kt} \quad \dots\dots\dots(4)$$

From Eq. (2) & Eq.(4) we have,

$$100 = ce^{k.0}$$

$$\therefore \; c = 100$$

From Eq. (3) & Eq.(4) we have,

$$50 = 100e^{30k}$$

$$or, \; e^{30k} = \frac{1}{2}$$

$$or, \; 30k = \ln(0.5)$$

$$or, \; k = \frac{1}{30}\ln(0.5)$$

$$\therefore \; k = -0.0231$$

Using the values of c & k in Eq.(4) we have,

$$p = 100e^{-0.0231t} \; \ldots\ldots\ldots(5)$$

This is the number of bacteria after t hr.

Again, For $t = 100$ years, we get from Eq. (5),

$$p = 100e^{-0.0231\times100}$$

$$\therefore \; p = 9.926mg$$

This amount of sample will remain after 100 years.

Again, let after $t = t_1$ years the amount of sample will remain 1mg. i.e, $p(t_1) = 1$.

Now from Eq. (4) we have,

$$1 = 100e^{-0.0231t_1}$$

$$or, \; e^{-0.0231t_1} = \frac{1}{40}$$

$$or, \; -0.0231T = \ln(0.025)$$

$$or, \; t_1 = -\frac{\ln(0.025)}{0.0231}$$

$$or,\ t_1 = 159.7 \text{ years}$$

This is the required time. (**ans.**)

Problem-04: When a cake is removed from an oven, it's temperature is measured at 300^0 F . Three minutes later it's temperature is 200^0 F. How long will it take for the cake to cool off to a room temperature of 70^0 F?

(a). Give a relation that gives the temperature of the cake after *t* mins.

(b). How long will it take for the cake to cool off to 75^0 F?

Solution: Let, $T(t)$ be the amount of temperature of the cake at any time *t* mins.

From Newton's Law of Cooling we know that the temperature of a body drops at a rate that is proportional to the difference between the temperature of the body and the temperature of the surrounding medium.

So $$\frac{dT}{dt} \propto \left(T - T_m\right)$$

$$or,\ \frac{dT}{dt} = k\left(T - T_m\right) \ \dots\dots\dots(1)$$

where, k is a proportional constant.

According to the question we have,

$$T\left(0\right) = 300 \ \dots\dots\dots (2)$$

$$T\left(3\right) = 200 \ \dots\dots\dots (3)$$

$$T_m = 70 \ \dots\dots\dots (4)$$

Now from Eq. (1) & Eq. (4) we can write,

$$\frac{dT}{dt} = k\left(T - 70\right)$$

$$or,\ \frac{dT}{T - 70} = kdt$$

$$or,\ \int \frac{dT}{T - 70} = k\int dt$$

$$or,\ \ln(T-70) = kt + \ln c$$

$$or,\ \ln(T-70) = \ln e^{kt} + \ln c$$

$$or,\ \ln(T-70) = \ln ce^{kt}$$

$$or,\ T-70 = ce^{kt}$$

$$\therefore\ T = 70 + ce^{kt}\ \ldots\ldots\ldots\ldots (5)$$

From Eq. (2) & Eq.(5) we have,

$$300 = 70 + ce^{k.0}$$

$$\therefore\ c = 230$$

From Eq. (3) & Eq.(5) we have,

$$200 = 70 + 230e^{3k}$$

$$or,\ e^{3k} = \frac{13}{23}$$

$$or,\ 3k = \ln\left(\frac{13}{23}\right)$$

$$or,\ k = \frac{1}{3}\ln\left(\frac{13}{23}\right)$$

$$\therefore\ k = -0.19018$$

Using the values of *c* & *k* in Eq.(5) we have,

$$T = 70 + 230e^{-0.19018t}\ \ldots\ldots\ldots(5)$$

This is the amount of temperature of the cake after *t* mins..

Again, let after $t = t_1$ mins. the amount of temperature of the cake will be 75^0 F. i.e, $T(t_1) = 75$.

Now from Eq. (5) we have,

$$75 = 70 + 230e^{-0.19018t_1}$$

$$or,\ e^{-0.19018t_1} = \frac{5}{230}$$

$$or,\ -0.19018t_1 = \ln\left(\frac{5}{230}\right)$$

$$or,\ t_1 = -\frac{1}{0.19018}\ln\left(\frac{5}{230}\right)$$

$$or,\ t_1 = 20.13 \text{ mins.}$$

This is the required time. (**ans.**)

Exercise:

Problem-01: The population of a certain community increases at a rate that is proportional to the number of people present at any time. If the population has doubled in 30 years, how long will it take to triple?

Problem-02: If a small metal bar, whose initial temperature is 20^0C, is dropped into a container of boiling water. How long will it take for the bar to reach 90^0C, if it is known that temperature increases 2^0C in 1second?