LINEAR PARTIAL DIFFERENTAIL EQUATIONS OF ORDER ONE

Linear Partial Differential Equations of Order One: A differential equation involving derivatives p and q only and no higher is called of order one. If, in addition, the degree or power of p and q is unity, then it is a linear partial differential equation of order one.

Example: 1. $3xp + 9yq = z$

2. $px^3 + qy^4 = z^2$

The standard form of linear partial differential equation of order one is,

$$Pp + Qq = R \qquad \cdots(A)$$

where, P, Q and R being functions of x, y and z. This is also known as Lagrange equation.

The general solution of (1) is,

$$\phi(u, v) = 0$$

where ϕ is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are solutions of equations,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \cdots(B)$$

which are called Lagrange auxiliary or subsidiary equations for (1).

Working procedure for solving $Pp + Qq = R$ by Lagrange's method:

Step-1: Put the given linear partial differential equation in the standard form,

$$Pp + Qq = R \qquad \cdots(A)$$

Step-2: Write down Lagrange's auxiliary equations for (1) namely,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \cdots(B)$$

Step-3: Solve (2) by well-known methods. Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step-4: The general solution (or integral) of (1) is then written in one of the following three equivalent forms:

$$\phi(u,v)=0\ ,\ u=\phi(v)\ \text{and}\ v=\phi(u).$$

Problem-01: Solve $(y^2z/x)p+xzq=y^2$

Solution: Given that, $(y^2z/x)p+xzq=y^2$ $\cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{(y^2z/x)}=\frac{dy}{xz}=\frac{dz}{y^2} \qquad \cdots(2)$$

Taking the first two fractions of (2), we get

$$\frac{dx}{(y^2z/x)}=\frac{dy}{xz}$$

$$or,\ \frac{xdx}{y^2z}=\frac{dy}{xz}$$

$$or,\ x^2dx=y^2dy$$

$$or,\ x^2dx-y^2dy=0$$

Integrating,

$$\frac{x^3}{3}-\frac{y^3}{3}=\frac{c_1}{3}$$

$$or,\ x^3-y^3=c_1 \qquad \cdots(3)$$

Next, taking the first and the last fractions of (2), we get

$$\frac{dx}{(y^2z/x)}=\frac{dz}{y^2}$$

$$or,\frac{xdx}{y^2z}=\frac{dz}{y^2}$$

$$or, xdx=zdz$$

$$or,\ xdx-zdz=0$$

Integrating,

$$\frac{x^2}{2}-\frac{z^2}{2}=\frac{c_2}{2}$$

$$or,\ x^2-z^2=c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$\phi\left(x^3-y^3, x^2-z^2\right)=0$$

where, ϕ is an arbitrary constant.

Problem-02: Solve $p\tan x+q\tan y=\tan z$

Solution: Given that, $p\tan x+q\tan y=\tan z \qquad \cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z} \qquad \cdots(2)$$

Taking the first two fractions of (2), we get

$$\frac{dx}{\tan x}=\frac{dy}{\tan y}$$

$$or,\ \cot x dx=\cot y dy$$

$$or,\ \cot x dx-\cot y dy=0$$

Integrating,

$$\ln(\sin x)-\ln(\sin y)=\ln c_1$$

$$or,\ \ln\left(\frac{\sin x}{\sin y}\right)=\ln c_1$$

$$or,\ \frac{\sin x}{\sin y}=c_1 \qquad \cdots(3)$$

Next, taking the last two fractions of (2), we get

$$\frac{dy}{\tan y}=\frac{dz}{\tan z}$$

$$or,\ \cot y dy = \cot z dz$$

$$or,\ \cot y dy - \cot z dz = 0$$

Integrating,

$$\ln(\sin y) - \ln(\sin z) = \ln c_2$$

$$or,\ \ln\left(\frac{\sin y}{\sin z}\right) = \ln c_2$$

$$or,\ \frac{\sin y}{\sin z} = c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$\frac{\sin x}{\sin y} = \phi\left(\frac{\sin y}{\sin z}\right)$$

where, ϕ is an arbitrary constant.

Problem-03: Solve $zp = -x$

Solution: Given that, $zp = -x \qquad \cdots(1)$

The Lagrange's auxiliary equations for (1) are,

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \qquad \cdots(2)$$

Taking the first and the last fractions of (2), we get

$$\frac{dx}{z} = \frac{dz}{-x}$$

$$or, -xdx = zdz$$

$$or, xdx + zdz = 0$$

Integrating,

$$\frac{x^2}{2} - \frac{y^3}{2} = \frac{c_1}{2}$$

$$or,\ x^2 + y^2 = c_1 \qquad \cdots(3)$$

Next, taking the last two fractions of (2), we get

$$\frac{dy}{0} = \frac{dz}{-x}$$

$$or,\ dy = 0$$

Integrating,

$$y = c_2 \qquad \cdots(4)$$

From (3) and (4) the required general solution (integral) is,

$$x^2 + y^2 = \phi(y)$$

where, ϕ is an arbitrary constant.

Exercise:

1. $2p + 3q = 1$
2. $yzp + 2xq = xy$
3. $x^2 p + y^2 q + z^2 = 0$
4. $xp + yq = z$