LINEAR PARTIAL DIFFERENTAIL EQUATIONS OF ORDER TWO

PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

INTRODUCTION: An equation is said to be of order two, if it involves at least one of the differential coefficients r = $(\partial^2 z / \partial^2 x)$, s = $(\partial^2 z / \partial x\, \partial y)$, t =$(\partial^2 z / \partial^2 y)$, but now of higher order; the quantities p and q may also enter into the equation. Thus the general form of a second order Partial differential equation is

$$f(\, x, y, z, p, q, r, s, t) \;=0 \qquad ...(1)$$

The most general linear partial differential equation of order two in two independent variables x and y with variable coefficients is of the form

$$Rr \;+\; Ss \;+\; Tt \;+Pp \;+Qq \;+Zz \;=\; F \qquad ...(2)$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

Ex.1: Solve $r \;=\; 6x$.

Sol. The given equation can be written as $\frac{\partial^2 z}{\partial x^2} \;=\; 6x$ ...(1)

Integrating (1) w. r. t. $x\frac{\partial z}{\partial x} \;=\; 3x^2 \;+\; \emptyset_1(y)$ ...(2)

where $\emptyset_1(y)$ is an arbitrary function of y.

Integrating (2) w. r. t. we get

$$x\, z \;=\; x^3 \;+\; x\, \emptyset_1(y) \;+\; \emptyset_2(y)$$

where $\emptyset_2$(y) is an arbitrary function of y.

Ex.2. $ar \;=\; xy$

Sol: Given equation can be written as $\frac{\partial^2 z}{\partial x^2} = \frac{1}{a}xy$ ...(1)

Integrating (1) w. r. t., x, we get

$$\frac{\partial z}{\partial x} = \left(\frac{y}{a}\right)\frac{x^2}{2} + \emptyset_1(y) \qquad ...(2)$$

where $\emptyset_1$(y) is an arbitrary function of y

Integrating (2) w. r. t., x,

$$z = \left(\frac{y}{a}\right)\frac{3}{6} + x\, \emptyset_1(y) + \emptyset_2(y)$$

Ex.3: Solve $r = 2y^2$

Sol: Try yourself.

Ex. 4. Solve $t = \sin(xy)$

Sol. Given equation can be written as $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$...(1)

Integrating (1) w. r. t., y

$$\frac{\partial z}{\partial y} = -\left(\frac{1}{x}\right)\cos(xy) + \emptyset_1(x) \qquad \ldots(2)$$

Integrating (2) w. r. t., y

$$z = -\left(\frac{1}{x^2}\right)\sin(xy) + y\,\emptyset_1(x) + \emptyset_2(x)$$

which is the required solution, $\emptyset_1$, $\emptyset_2$ being arbitrary functions.

Exercises: $xys = 1$

Sol: We know that $s = \frac{\partial^2 z}{\partial x \partial y}$

Therefore $xy\frac{\partial^2 z}{\partial x \partial y} = 1$

or $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{xy}$

Integrating w.r.t., y we have

$$\frac{\partial z}{\partial x} = \frac{1}{x}\log y + f(x)$$

Again integrating w.r.t., x we get

$$z = \log x \log y + \int f(x)\,dx + F(y)$$

0r $$z = \log x \log y + g(x) + F(y)$$

Exercises: $2x + 2y = s$

Sol: The given equation can be written as

$$\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$$

Integrating w.r.t., y, we have

$$\frac{\partial z}{\partial x} = y^2 + 2xy + f(x)$$

Integrating w.r.t., x, we have

$$z = y^2 x + x^2 y + \int f(x)\,dx + F(y)$$

$$\therefore z = y^2 x + x^2 y + g(x) + F(y)$$

Exercises: $xr + p = 9x^2 y^3$

Sol: The given equation can be written as

$$x\frac{\partial^2 z}{\partial x^2} + p = 9x^2 y^3$$

$$\Rightarrow x\frac{\partial p}{\partial x} + p = 9x^2 y^3$$

$\Rightarrow \frac{\partial p}{\partial x} + \frac{p}{x} = 9xy^3$ … (1)

which is linear first order differential equation in p

$\therefore$ I. F. is $e^{\log x} = x$

Multiplying (1) by x we get

$$x\left[\frac{\partial p}{\partial x} + \frac{p}{x}\right] = 9x^2 y^3$$

$$\Rightarrow px = 9\int x^2 y^3\, dx$$

$$\Rightarrow px = 9\frac{x^3y^3}{3} + f(y)$$

$$\Rightarrow px = 3x^3y^3 + f(y)$$

$$\Rightarrow p = \frac{3x^3y^3 + f(y)}{x}$$

$$\Rightarrow \frac{\partial z}{\partial x} = 3x^2y^3 + \frac{f(y)}{x}$$

Integrating with respect to x we get

$$z = x^3y^3 + f(y)\log x + F(y)$$

Exercise: Find the surface satisfying $r + s = 0$, and touching the elliptic paraboloid $z = 4x^2 + y^2$ along the surface of plane $y = 2x + 1$.

Sol: From the given equation we have $\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 0$.

Integrating with respect to x, we have

$$p + q = f(y)$$

Now, the auxiliary system is

$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(y)}$ ...(1)

Taking first two fractions we get

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrating we get

$$x = y + a$$

$$\Rightarrow \qquad x - y = a \qquad \ldots(2)$$

Also from 2nd and 3rd fractions of (1), we get

$$\frac{dy}{1} = \frac{dz}{f(y)}$$

$$\Rightarrow dz = f(y)dy$$

$$\Rightarrow z = \varphi(y) + b$$

or $\qquad z = \varphi(y) + F(a)$

$$\Rightarrow z = \varphi(y) + F(x - y) \qquad \ldots (3)$$

From (3), we get

$$p = \frac{\partial z}{\partial x} = F'(x - y) \qquad \ldots(4)$$

$$q = \frac{\partial z}{\partial y} = \varphi'(y) - F'(x - y) \qquad \ldots (5)$$

Since $\quad z = 4x^2 + y^2$

$$\therefore p = \frac{\partial z}{\partial x} = 8x \qquad \ldots(6)$$

& $\qquad q = \frac{\partial z}{\partial y} = 2y \qquad \ldots(7)$

From (4) and (6)

$$F'(x - y) = 8x \qquad \ldots (8)$$

From (5) and (7)

$$\varphi'(y) - F'(x - y) = 2y \qquad \ldots (9)$$

Adding (8) and (9) we get

$$\varphi'(y) = 8x + 2y$$

$$= \frac{8}{2}(y - 1) + 2y$$

$$= 6y - 4$$

Integrating w. r. t., y, we get

$$\varphi(y) = 3y^2 - 4y + b \qquad \dots (10)$$

Also, from (8)

$$-F'(x-y) = 8x = -8(y-x-1) = 8(x-y+1)$$

Integrating w. r. t., $(x-y)$ we get

$$-F(x-y) = 4(x-y)^2 + 8(x-y) + c \qquad \dots (11)$$

Substituting (10) and (11) in (3) we get

$$z = 3y^2 - 4y + b - 4(x-y)^2 - 8(x-y) + c$$

$$= -4x^2 - y^2 + 4y - 8x + 8xy + d$$

From the given condition,

$$4x^2 + (2x+1)^2 = -4x^2 - (2x+1)^2 + 4(2x+1) - 8x + 8x(2x+1) + d$$

$$\Rightarrow 8x^2 + 2(2x+1)^2 = 4(2x+1) - 8x + 8x(2x+1) + d$$

$$\Rightarrow 8x^2 + 8x^2 + 2 + 8x = 8x + 4 - 8x + 16x^2 + 8x + d$$

$$\Rightarrow d = -2$$

Therefore $\qquad z = -4x^2 - y^2 + 4y - 8x + 8xy - 2$

which is required surface.