APPLICATION OF PARTIAL DIFFERENTIAL EQUATION

Classification of second order partial differential equation

Definition: A second order partial differential equation which is linear w. r. t., the second order partial derivatives i.e. r, s and t is said to be a quasi linear PDE of second order. For example the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \ldots.(1)$$

where $f(x, y, z, p, q)$ need not be linear, is a quasi linear partial differential equation. Here the coefficients R, S, T may be functions of x and y, however for the sake of simplicity we assume them to be constants.

The equation (1) is said to be

(i) Elliptic if $S^2 - 4RT < 0$
(ii) Parabolic if $S^2 - 4RT = 0$
(iii) Hyperbolic if $S^2 - 4RT > 0$

BOUNDARY VALUE PROBLEMS: The function v in addition to satisfying the Laplace and Poisson equations in bounded region R in three dimensional space, should also satisfy certain boundary conditions on the boundary C of this region. Such problems are referred to as Boundary value problems for Laplace and poison equations. If a function $f \in C^n$, then all its derivatives of order n are continuous. If $f \in C^0$,then we mean that f is continuous.

There are mainly three types of boundary value problems for Laplace equation. If $f \in C^0$, and is prescribed on the boundary C of some finite region R, the problem of determining a function $\emptyset(x, y, z)$ such that $\nabla^2\emptyset = 0$ within R and satisfying $\emptyset = f$ on C,is called the boundary value problem of first kind or Dirichlet problem. The second type of boundary value problem (BVP) is to determine the function $\emptyset(x, y, z)$ so that $\nabla^2\emptyset = 0$ within R while $\frac{\partial \emptyset}{\partial n}$ is sepecified at every point of C, where $\frac{\partial \emptyset}{\partial n}$ is the normal derivative of $\emptyset$. This problem is called the Neumann problem.

The third type of boundary value problem is concerned with the determination of the function $\emptyset(x, y, z)$ such that $\nabla^2\emptyset = 0$ within R,while a boundary condition of the form $\frac{\partial \emptyset}{\partial n} + h\,\emptyset = f$,where $h \geq 0$ is specified at every point of the boundary C.This is called a mixed boundary value problem or Churchill's problem.

PARABOLIC DIFFERENTIAL EQUATIONS

The have equation of the form $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ with $S^2 - 4RT = 0$ is known as parabolic differential equation. The diffusion phenomenon such as conduction heat in solids and diffusion of viscous fluid flow as generated by a PDE of parabolic type.

The general equation for heat transfer is governed by the following equations

$$\frac{\partial T}{\partial t} = k\nabla^2 \mathrm{T}$$

Where $\frac{\partial T}{\partial t}$ is the time derivative and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ represents the derivative w.r.t., space.

Heat Equation: The heat conduction equation $\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$

EX: The ends A and B of a rod, $10\ cm$ in length are kept at temperature 0^0c and 100^0 c respectively until the steady state conditions prevails. Suddenly the temperature at the end A is increased to 20^0 c and the end B is decreased to 60^0 c. Find the temperature distribution in rod at time at t.

Sol. The problem is described by

$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$; 0<x<10

Subject to the conditions

$$T\ (0, t) = 10$$

$$T(10, t) = 100$$

For steady state $\frac{d^2 T}{dx^2} = 0$

Which implies that $T_s = Ax + B$

Now for x = 0, T= 0 implies that B = 0, Therefore $T_s = Ax$

And for x = 0, T = 100^0 C, implies that A = 10

Thus the initial steady temperature distribution in rod is

$T_s(x) = 10x$

Similarly when the temperature at the ends A and B are changed to 20^0c and 60^0C, the final steady temperature in rod is

$T_s(x) = 4x + 20$

Which will be attained after long time. At any instant of time the temperature

$T\ (x, t)$ in rod is given by

$$T(x,t) = T_t(x,t) + T_s(x)$$

Where $T_t(x,t)$ is the transient temperature distribution which tends to zero as $t \to \infty$. Now $T(x,t)$ satisfies the given partial differential equation. Hence its general solution is of the form

$$T(x,t) = T_t(x,t) + T_s(x)$$

$$T(x,t) = 4x + 20 + e^{-K\lambda^2 t}\,(B\cos\lambda x + C\sin\lambda x)$$

For x = 0, T = 20^0C, we obtain

$20 = 20 + B\ e^{-K\lambda^2 t} \Rightarrow B = 0,\ \ t > 0$

For x = 10, T = 60^0 we get

$60 = 60 + e^{-K\lambda^2 t} C\ \sin 10\,\lambda$

$\Rightarrow \sin 10\,\lambda = 0 \quad \Rightarrow \quad \lambda = \frac{n\pi}{10}, \quad n \in I$

The principle of superposition yields

$$T(x,t) = 4x + 20 + \sum_{n=1}^{\infty} C_n \exp\left[-k\left(\frac{n\pi}{10}\right)^2 t\right]\sin\left(\frac{n\pi}{10}x\right)$$

using the initial condition $T = 10x$, when t= 0, we obtain

$$10x = 4x + 20 + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{10}x\right)$$

Where $C_n = \frac{2}{10}\int_0^{10}(6x-20)\sin\left(\frac{n\pi}{10}x\right)dx$

$$= \frac{-1}{5}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]$$

Thus the required solution is

$$T(x,t) = 4x + 20 - \frac{1}{5}\sum_{n=1}^{\infty}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]\exp\left[-k\left(\frac{n\pi}{10}\right)^2 t\right]\sin\left(\frac{n\pi}{10}x\right)$$

HYPERBOLIC DIFFERENTIAL EQUATIONS:

One of the most important and typical homogenous hyperbolic differential equation is the wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

Where C is the wave speed.

The differential equation above is used in many branches of physics and engineering and is seen in many situations such as transverse vibrations in strings or membrane, longitudinal vibrations in a bar, propagation of sound waves, electromagnetic waves, sea waves, elastic waves in solid and surface waves in earth quakes.

The solution of wave equations are called wave functions.

Remark: The Maxwell's equations of electromagnetic theory is given by

$$\nabla.E = 4\pi P$$

$$\nabla.\vec{H} = 0$$

$$\nabla\times\vec{E} = \frac{-1}{C}\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H} = \frac{4\pi i}{C} + \frac{1}{C}\frac{\partial\vec{t}}{\partial t}$$

Where E is an electric field, ρ is electric charge density, H is the magnetic field, C is the current density and C is the velocity of light.

Exercise : show that in the absence of a charge, the electric field and the magnetic field in the Maxwell's equation satisfy the wave equation.

Solution: we have

Curl $\vec{E}$ = $\nabla\times\vec{E} = \frac{-1}{C}\frac{\partial\vec{H}}{\partial t}$

Consider $\nabla\times(\nabla\times E) = \nabla\times\left(\frac{-1}{C}\frac{\partial\vec{H}}{\partial t}\right)$

$$= \frac{-1}{C}\frac{\partial}{\partial t}\left(\nabla\times\vec{H}\right)$$

This implies $\nabla\times(\nabla\times E) = \frac{-1}{C^2}\frac{\partial^2 E}{\partial t^2}$

But $\nabla\times(\nabla\times E)$ Can be expressed as

$$\nabla(\nabla\cdot E) - \nabla^2 E = \nabla(4\pi\rho) - \nabla^2 E$$

$$= -\nabla^2 E$$

$$-\nabla^2 E = \frac{-1}{C^2}\frac{\partial^2 E}{\partial t^2}$$

Or $\frac{\partial^2 E}{\partial t^2} = C^2 \nabla^2 E$

Which is a wave equation satisfied by $\vec{E}$

Similarly we can observe the magnetic field H satisfies the wave equation

$\frac{\partial^2 H}{\partial t^2} = c^2 \nabla^2 H$.

Solution of Wave equation: (Method of separation of variables)

We have $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Let U(x, t) = X(x) T(x) be the solution of (1)

$\therefore \frac{\partial^2 U}{\partial t^2} = XT''$ and $\frac{\partial^2 U}{\partial x^2} = TX''$

Using in (1),we get

$XT'' = C^2 TX'' \Rightarrow \frac{T''}{T} = C^2 \frac{X''}{X} = \lambda$ (say)

Where λ is a separation parameter

$$\Rightarrow T'' - \lambda T = 0 \quad ...(2)$$

And $$C^2 X'' - \lambda X = 0 \quad ...(3)$$

T = $Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$

CASE I: If $\lambda > 0$ say $\lambda = k^2$

T = $Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$

CASE I: If $\lambda > 0$ say $\lambda = k^2$

Therefore $T = Ae^{kt} + Be^{-kt}$ ...(4)

Similarly

$$C^2X'' - K^2X = 0$$

$$X(x) = De^{\frac{K}{C}x} + Ee^{-\frac{K}{C}x}$$

$$\therefore U(x,t) = (Ae^{Kt} + Be^{-Kt})(De^{\frac{K}{C}x} + Ee^{-\frac{K}{C}x})$$

This is the required solution

Case II: if $\lambda = 0$

Then $T'' = 0$ and $C^2X'' = 0$

$\Rightarrow T = Ct + D$ and $X = Ax + B$

Therefore $U(x,t) = (Ax + B)(Ct + D)$

Case III: If $\lambda < 0$ say $\lambda = -K^2$

$$\Rightarrow \frac{T''}{T} = -K^2 \text{ and } \frac{C^2X''}{X} = -K^2$$

$\Rightarrow T'' + K^2T = 0$ and $C^2X'' + K^2X = 0$

So T = $(A\cos Kt + B\sin Kt)$ and X = $\left(D\cos\frac{K}{C}x + E\sin\frac{K}{C}x\right)$

Therefore U(x, t) = $(A\cos Kt + B\sin Kt)\left(D\cos\frac{K}{C}x + E\sin\frac{K}{C}x\right)$