

**Chapter 07**

**Equation and Its Geometry**

**Introduction:**

**General Equation of First (1st ) Degree:**

The general form of the first-degree equation is, . It always represents a straight line in-plane, provided at least  or, .

**Locus of a Point:**

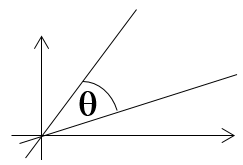
A locus of a point is a path in which it moves in a plane or in space by following the certain rules/conditions.

**Pair of Straight Lines:**

A pair of straight lines is the locus of a point whose coordinates satisfy a second-degree equation

. A collection of combined two straight lines is called a pair of straight lines.

**Homogeneous Equation:**

An equation in which the degree of each term in it is equal is called Homogeneous equation. Such as  is a homogeneous equation of degree or order 2 because the degree of its every term is two. It is noted that the homogeneous equation always represents straight lines passing through the origin.

**Theorem:** Find the angle between the straight lines represented by the homogeneous equation.

**Proof:** Given the homogeneous equation of second degree is, 

Suppose  and  are the two straight lines represented by the given homogeneous equation.

So, 





This equation is the same as to the given equations, so the ratio of the coefficient of like terms is equal.

Now comparing the coefficients



From 2nd and 3rd parts we get 

From 1nd and 3rd parts we get 

If  be the angle between two straight lines represented by the given equation then













***NOTE:*** The angle between the lines represented by the equation  is calculated by the formula .

* Lines be perpendicular if 
* Lines be parallel/coincident if 
* Since two lines pass through the point , so lines must be coincident.
* Lines represented by the homogeneous equation is real if .
* Lines represented by homogeneous equation are imaginary if , but passes through the point .
* The equation of the bisectors of an angle produced by the pair of straight-line represented by the equation  is .

**Example:** Find the lines represented by the equation .

**Solution: (**1st Process) Given the homogeneous equation is as follows 

We expressed the given equation as, 







Therefore  and .

These are the straight lines passing through the origin.

(2nd Process) Given the homogeneous equation is as follows 

We expressed the given equation as, 













Taking a positive sign we get, 

Therefore, 

And taking negative sign we get, 

Therefore, 

Therefore  and  are the straight lines passing through the origin.

**Example:** Find the angle between the lines represented by the equation .

**Solution:** Given the homogeneous equation is as follows, 



Comparing this equation with the general homogeneous equation  we have  and .

Let an angle between the lines is .

Then we have, 









Therefore the angle between the lines is.

**Example:** Find the equation of the bisectors of an angle produced by the pair of straight-line represented by the equation at the origin.

**Solution:** Given the homogeneous equation is as follows 

Comparing the given equation with the general homogeneous equation  we have  and .

We know that the equation of the bisectors of an angle produced by the pair of straight-line represented by the equation  is .

So the required equation is, 









**Non-Homogeneous Equation:**

An equation in which the degree of each term in it is not equal is called Non-homogeneous equation. Such as  is a non- homogeneous equation of degree or order 2.

We know the general equation of the second degree is . Descartes found that the graphs of second-degree equations in two variables in-plane always fall into one of seven categories: [1] single point, [2] pair of straight lines, [3] circle, [4] parabola, [5] ellipse, [6] hyperbola, and [7] no graph at all.

**Theorem:** Find the condition that the general equation of second degree may represent two straight lines.

Proof: Given the general equation of second degree is, 

Express the above equation as a quadratic equation in  as 

Solving we get, 











Equation (i) represents two straight lines if it is possible to factorize the left-hand side of (i) as a product of two linear factors. It will be done if the quantity of under the square root sign in the equation (ii) be a perfect square, that means  must be a perfect square.

So that  will be a perfect square if the roots of the equation is equal.

Now, 







Roots of the above quadratic equation in y be equal if the discriminant of the equation .

Here, 















 This is the required condition to represents the straight lines.

***NOTE:***

Rule 1: General equation of second degree, , represents a pair of straight lines if, 

Rule 2: Angle between the lines represented by the equation, is calculated by the formula 

**θ**

* Lines be perpendicular if 
* Lines be parallel if 
* Lines represented by Non-homogeneous equation is real if .
* Lines represented by Non-homogeneous equation are imaginary if . But passes through a real point.

Rule 3: If  then the general equation of second degree,  will represent a conic,

* A circle if 
* A parabola if 
* An ellipse if 
* A hyperbola if 
* A rectangular hyperbola if 

From the above condition, we may construct the below table for our better understanding-

|  |  |  |  |
| --- | --- | --- | --- |
| General equation of second degree is, , ……….. *(A)*  The equation *A* may represent a pair of straight line or conic, it depends on the value of  .  where, | | | |
| If , then the equation (*A*) will represent a pair of a straight line | | | |
| Condition | **Name of Geometry** | **General Equation** | **Figure** |
| if | Lines be perpendicular |  |  |
| if | Lines be parallel /coincident |  |  |
| If , then the equation (*A*) will represent a conic | | | |
| if | Circle |  |  |
| if | Parabola |  |  |
| if | Ellipse |  |  |
| if | Hyperbola |  |  |
|  | Rectangular Hyperbola |  |  |

**The Intersection of the Lines:**

The equation of the bisectors of an angle produced by the pair of straight-line represented by the equation  is , where  is the intersection point of those lines.

Determination of intersecting point  of the straight lines represented by the equation :

Let 

Then set 



Now calculate the intersecting point of the equation and we get the intersecting point  of the straight lines represented by the equation .

**Example:** Show that  represents a pair of straight lines.

**Solution:** Given equation is, 

We may write the given equation as  …………. (1)

Comparing equation (1) with the standard equation we get



Now, 









Since  so the given equation represents a pair of straight lines.

**Example:** Test the nature of the equation .

**Solution:** Given that, 

We may write the given equation as  ………. (1)

Comparing equation (1) with the standard equation we get



Now, 



Since  so the given equation represents a conic.

Again, 

Since  so the given equation represents a hyperbola.

**Example:** Test the nature of equation  and also find its centre.

**Solution:** (1st part) Given that, **

We may write the given equation as  …………. (1)

Comparing equation (1) with the standard equation we get



Now, 



Since  so the given equation represents a conic.

Again,  and 

Since  so the given equation represents a rectangular hyperbola.

(2nd part) Let, 

 and 

The centre of the conic is the intersection of two lines,





Solving  and  we have, 

Hence the centre is 

Reduction of equation to a standard form

󠆻󠆻 *Problem-03:*

Reduce the equation to the standard form.

Solution:

Given general equation of second degree is



Comparing this above equation with the standard equation we get



Now,









And



Since  and . So the equation represents an ellipse.

Suppose 

Now, Differentiating the function with respect to x and y partially and equating with zero, we get



And



Solving equation (i) and (ii) we get centre of the conic represented by the given equation.

Using cross multiplication method on equation (i) and (ii)







Therefore, the coordinates of centre is .

Therefore, the equation of the conic referred to centre as origin is



Where 

So the equation becomes 

When the xy term is removed by the rotation of axes then the reduced equation is



Then by invariants we have









We know,











Solving equations and we have  and 

The equation becomes 



Which is required equations.

Example: Reduce the equation  to the standard form.

Solution:

Given general equation of second degree is



Comparing this above equation with the standard equation we get



Now,



And



Since  and . So the equation represents hyperbola.

Suppose 

Now, Differentiating the function with respect to x and y partially and equating with zero, we get



And



Solving equation (ii) and (iii) we get centre of the conic represented by the given equation.

Using cross multiplication method on equation (ii) and (iii)







Therefore, the coordinates of centre is .

Therefore, the equation of the conic referred to centre as origin is



Where 

So the equation becomes 

When the xy term is removed by the rotation of axes then the reduced equation is



Then by invariants we have









We know,









Solving equations and we have  and 

The equation becomes 



Which is required equations. (As desired)

**Miscellaneous Problem**

For what value of  the equation represents a pair of straight lines.

Solution:

Given equation is,



Comparing this above equation with the standard equation we get



Here the given equation represents a pair of straight lines if .

Now,























 , This is the required value of. (Ans)

Find the equation of the straight lines represented by the equation

.

Solution:

Given equation is,



Arrange the above equation as a quadratic equation in x we get





















Taking positive we get 











Taking negative we get 











Therefore, required equations of the straight lines  and . (As desired)

Find the point of intersection of the straight lines represented by the equation

.

Solution:

Given equation is,



Suppose 

Now, Differentiating the function with respect to x and y partially and equating with zero, we get



And



Solving equation (i) and (ii) we get the point of intersection of lines represented by the given equation.

Using cross multiplication method on equation (i) and (ii)







Therefore, the coordinates of point of intersection is .

Find the angle between the straight lines represented by the equation

.

Solution:

Given equation is,



Comparing this above equation with the standard equation we get



Assume that be the angle between the straight lines then we have the followings











 (As desired)

Find the equation of the bisectors of the angle between the straight lines represented by the equation

.

Solution:

Given equation is,



Comparing this above equation with the standard equation we get



Suppose 

Now, Differentiating the function with respect to x and y partially and equating with zero, we get



And



Solving equation (i) and (ii) we get the point of intersection of lines represented by the given equation.

Using cross multiplication method on equation (i) and (ii)







Therefore, the coordinates of point of intersection is .

If be the point of intersection of lines by given equations the equation of the bisectors is as follows



















 (As desired)

**Exercise**

Theoretical Questions:

MCQ (Short) Questions:

Mathematical Problems (Broad Questions):

Practice: Find the nature of the following equations,

(i)  (ii) 