# FIXED POINT ITERATION METHOD

**Fixed point :** A point, say, **s** is called a fixed point if it satisfies the equation **x = g(x)**.

**Fixed point Iteration :** The transcendental equation **f(x) = 0** can be converted algebraically into the form **x = g(x)** and then using the iterative scheme with the recursive relation

$$x_{i+1} = g(x_i), \qquad i = 0, 1, 2, \ldots,$$

with some initial guess $x_0$ is called the fixed point iterative scheme.

**Algorithm - Fixed Point Iteration Scheme**

Given an equation f(x) = 0
Convert f(x) = 0 into the form x = g(x)
Let the initial guess be $x_0$
Do
    $x_{i+1} = g(x_i)$
while (none of the convergence criterion C1 or C2 is met)

- C1. Fixing apriori the total number of iterations **N** .
- C2. By testing the condition $| x_{i+1} - g(x_i) |$ (where **i** is the iteration number) less than some tolerance limit, say epsilon, fixed apriori.

**Numerical Example :**

Find a root of $x^4 - x - 10 = 0$ [ *Graph*]
Consider $g1(x) = 10 / (x^3 - 1)$ and the fixed point iterative scheme $x_{i+1} = 10 / (x_i^3 - 1)$, $i = 0, 1, 2, \ldots$ let the initial guess $x_0$ be **2.0**

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 2 | 1.429 | 5.214 | 0.071 | -10.004 | -9.978E-3 | -10 | -9.99E-3 | -10 |

So the iterative process with **g1** gone into an infinite loop without converging.

Consider another function $g2(x) = (x + 10)^{1/4}$ and the fixed point iterative scheme $x_{i+1} = (x_i + 10)^{1/4}$, $i = 0, 1, 2, \ldots$

let the initial guess $x_0$ be **1.0, 2.0 and 4.0**

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $x_i$ | 1.0 | 1.82116 | 1.85424 | 1.85553 | 1.85558 | **1.85558** | |
| $x_i$ | 2.0 | 1.861 | 1.8558 | 1.85559 | 1.85558 | **1.85558** | |
| $x_i$ | 4.0 | 1.93434 | 1.85866 | 1.8557 | 1.85559 | 1.85558 | **1.85558** |

That is for **g2** the iterative process is converging to **1.85558 with** any initial guess.

Consider $\mathbf{g3(x) = (x+10)^{1/2}/x}$ and the fixed point iterative scheme

$$x_{i+1} = (x_i + 10)^{1/2} / x_i, \quad i = 0, 1, 2, \ldots$$

let the initial guess $\mathbf{x_0}$ be **1.8,**

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | . . . | 98 |
|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 1.8 | 1.9084 | 1.80825 | 1.90035 | 1.81529 | 1,89355 | 1.82129 | . . . | **1.8555** |

That is for g3 with any initial guess the iterative process is converging but very slowly to

Geometric interpretation of convergence with g1, g2 and g3

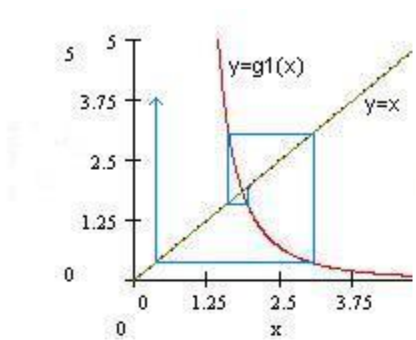


Fig g1

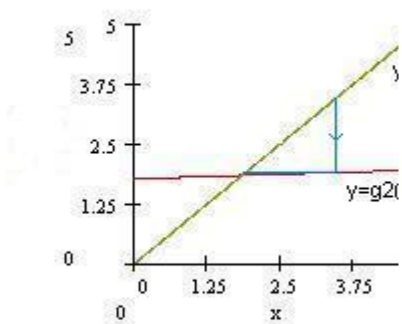


Fig g2

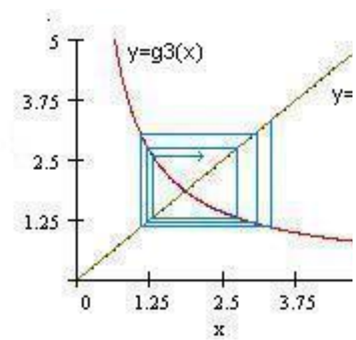


Fig g3

The graphs Figures Fig g1, Fig g2 and Fig g3 demonstrates the Fixed point Iterative Scheme with g1, g2 and g3 respectively for some initial approximations. It's clear from the

- Fig g1, the iterative process does not converge for any initial approximation.
- Fig g2, the iterative process converges very quickly to the root which is the intersection point of y = x and y = g2(x) as shown in the figure.
- Fig g3, the iterative process converges but very slowly.

**Example 2** **:** The equation $x^4 + x = \in$, where $\in$ is a small number , has a root which is close to $\in$. Computation of this root is done by the expression $\xi = \in - \in^4 + 4\in^7$ Then find an iterative formula of the form $x_{n+1} = g(x_n)$, if we start with $x_0 = 0$ for the computation then show that we get the expression given above as a solution. Also find the error in the approximation in the interval **[0, 0.2]**.

**Proof**

Given $x^4 + x = \in$

$x(x^3 + 1) = \in$

$x = \in/(1 + x^3)$ or $x_i = \in/(1 + x_i^3)$ $\quad i = 0, 1, 2, \ldots$

$x_0 = 0$

$x_1 = \in$

$$
\begin{aligned}
x_2 &= \in/(1 + \in_i^3) = \in(1 + \in_i^3)^{-1} \\
&= \in(1 - \in^3 + \in^6 + \ldots) \\
&= \in - \in^4 + \in^7 + \ldots
\end{aligned}
$$

$x_3 = \in/(1 + (\in - \in^4 + \in^7)^3) = \in[1 + (\in - \in^4 + \in^7)^{-3}] = \in - \in^4 + 4\in^7$

Now taking $\xi = \in - \in^4 + 4\in^7$

$$
\begin{aligned}
\text{error} &= \xi^4 + \xi - \in \\
&= (\in - \in^4 + 4\in^7)^4 + (\in - \in^4 + 4\in^7) - \in \\
&= 22\in^{10} + \textbf{higher order power of } \in
\end{aligned}
$$

**Condition for Convergence** **:**

If **g(x)** and **g'(x)** are continuous on an interval **J** about their root **s** of the equation $x = g(x)$, and if $|g'(x)| < 1$ for all **x** in the interval **J** then the fixed point iterative process $x_{i+1} = g(x_i)$, $i = 0, 1, 2, \ldots$, will converge to the root $x = s$ for any initial approximation $x_0$ belongs to the interval **J** .