8.2.5 Maxima and Minima of a Tabulated Function:

Newton’s forward interpolation formula for the function $y = f(x)$ is given by

$$
y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \cdots,
$$

$$
p = \frac{x-x_0}{h}
$$

... (1)

Differentiating (1) with respect to $p$

$$
\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 + \cdots
$$

... (2)

For finding maxima/minima of a function $y = f(x)$, $\frac{dy}{dx} = \frac{dy}{dp} \frac{dp}{dx} = 0$

$$
\frac{dp}{dx} = \frac{1}{h} \neq 0, \quad \therefore \frac{dy}{dp} = 0
$$

... (3)

Neglecting $4^{th}$ and higher order differences in equation (2) and substituting in (3),
we get a quadratic equation of the form $A + Bp + Cp^2 = 0$, where $A, B, C$ are constants. Solving for $p$ and substituting in $x = x_0 + ph$, we get points of
maxima/minima for the function \( y = f(x) \).

- Newton’s forward method is apt for finding extreme values of a tabulated data, wherever their location may be, by index \( p \) assuming values \( |p| \geq 1 \), if the extreme value is not in vicinity of the point \( (x_0, y_0) \). Yet we may also use Newton’s backward or Stirling’s central differences formulae to locate extreme values, if desired.

**Example 7** From the following data, find maximum and minimum values of \( y \).

\[
\begin{array}{cccc}
 x & 0 & 2 & 4 & 6 \\
 f(x) & 2 & 0 & -50 & -196 \\
\end{array}
\]

**Solution:** Constructing forward difference table for the function \( y = f(x) \)

\[
\begin{array}{cccc}
 x & y & \Delta & \Delta^2 & \Delta^3 \\
 0 & 2 & & & \\
 2 & 0 & -2 & 48 & 48 \\
 4 & -50 & -2 & -96 & \\
 6 & -196 & & & \\
\end{array}
\]

Newton’s forward interpolation formula for the function \( y = f(x) \) is given by

\[
y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots, \quad p = \frac{x-x_0}{h}, \quad \cdots (1)
\]

Taking \( x_0 = 0, \ y_0 = 2, \ \Delta y_0 = -2, \ \Delta^2 y_0 = -48, \ \Delta^3 y_0 = -48 \)
Substituting these values in (1), we get
\[ y \equiv 2 + p(-2) + \frac{p(p-1)}{2}(-48) + \frac{p(p-1)(p-2)}{6}(-48) \]

\[ \Rightarrow y \equiv 2 - 2p - 24(p^2 - p) - 8(p^3 - 3p^2 + 2p) \]
\[ \Rightarrow y \equiv -8p^3 + 6p + 2 \quad \ldots (2) \]

Differentiating (2) with respect to \( p \), we get
\[ \frac{dy}{dp} = -24p^2 + 6 \]

For \( y \) to be maximum, \( \frac{dy}{dp} = 0 \)
\[ \Rightarrow -24p^2 + 6 = 0 \]
\[ \Rightarrow p = 0.5, -0.5 \]

Substituting in (2), maximum and minimum values of \( y \) are 4 and 0 respectively.

**Example 8** From the following table, find \( x \) for which \( y \) is maximum.

<table>
<thead>
<tr>
<th>( x )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.205</td>
<td>0.240</td>
<td>0.259</td>
<td>0.262</td>
<td>0.250</td>
<td>0.224</td>
</tr>
</tbody>
</table>
Also find maximum value of $y$.

**Solution:** Constructing forward difference table for the function $y = f(x)$, up to third differences

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = f(x)$</th>
<th>$\Delta$</th>
<th>$\Delta^2$</th>
<th>$\Delta^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.205</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.240</td>
<td>0.035</td>
<td>-0.016</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.259</td>
<td>0.019</td>
<td>-0.016</td>
<td>0.001</td>
</tr>
<tr>
<td>6</td>
<td>0.262</td>
<td>0.003</td>
<td>-0.015</td>
<td>0.001</td>
</tr>
<tr>
<td>7</td>
<td>0.250</td>
<td>-0.012</td>
<td>-0.014</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.224</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Newton’s forward interpolation formula for the function $y = f(x)$ is given by

$$y \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \cdots, \quad p = \frac{x-x_0}{h} \quad ... (1)$$

Taking $x_0 = 3$, $y_0 = 0.205$, $\Delta y_0 = 0.035$, $\Delta^2 y_0 = -0.016$, $\Delta^3 y_0 = 0$

Substituting these values in (1), we get

$$y \equiv (0.205) + p(0.035) + \frac{p(p-1)}{2} (-0.016) + 0 \quad ... (2)$$

Differentiating with respect to $p$, we get

$$\frac{dy}{dp} = 0.035 + \frac{2p-1}{2} (-0.016) = 0.035 - (0.008)(2p - 1)$$

For $y$ to be maximum, $\frac{dy}{dp} = 0$

$$\Rightarrow 0.035 - (0.008)(2p - 1) = 0$$

$$\Rightarrow p = 2.6875$$

Also $p = \frac{x-x_0}{h}$ or $x = x_0 + ph$

$$\Rightarrow x = 3 + 2.6875(1) = 5.6875$$

$\therefore y$ is maximum when $x = 5.6875$ or $p = 2.6875$

Substituting in (2), maximum value of $y$ is given by

$$y \equiv (0.205) + (2.6875)(0.035) + \frac{(2.6875)(2.6875-1)}{2} (-0.016) = 0.2628$$