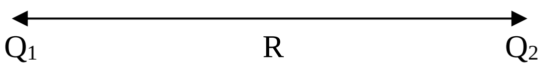
CHAPTER-02

Coulomb’s Law and Electric Field Intensity

Coulomb’s law

Coulomb’s law states that the force F between two point charges Q_1 and Q_2 along the line joining them is-

- Directly proportional to the product Q_1Q_2 of the charges
- Inversely proportional to the square of the distance R between



$$F \alpha Q_1$$

$$F \alpha Q_2$$

$$F \alpha \frac{1}{r^2}$$

Now we can easily write an equation based on above equations.

$$F \alpha \frac{Q_1 Q_2}{R^2} a_R$$

$$F = K \frac{Q_1 Q_2}{R^2} a_R \qquad \text{[Where } K \text{ is proportional constant]}$$

Where,

The value of, $K = \frac{1}{4\pi\epsilon_0}$

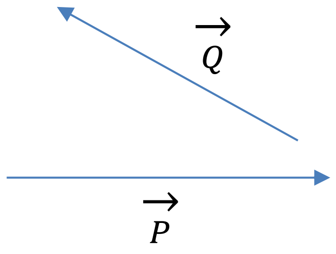
$$\therefore F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} a_R \qquad [\text{Where, } \epsilon_0 = 8.854 \times 10^{-12} = \frac{1}{36\pi} \times 10^{-9} F/m]$$

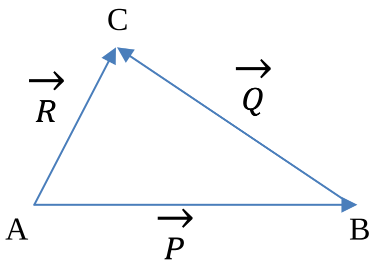
The new constant ϵ_0 is called the permittivity of free space and has the magnitude, measured in farads per meter (F/m). This is a mathematical quantity that represents, how much electric field is permitted (Penetrated) in free space or vacuum. It is an ideal physical constant that represents the absolute dielectric permittivity of a vacuum. In other words, ϵ_0 quantifies the ability of a vacuum to allow electric field lines to flow through.

In SI units, Q_1 and Q_2 are measured in coulombs (C), the distance R is in meters (m) and the force F is in newtons (N).

Triangle law of forces

If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order, the third side of the triangle represents their resultant in magnitude and direction in reverse order.





- $\vec{P}$ and $\vec{Q}$ are two non-zero forces
- These two are represented by two sides of a triangle AB, BC respectively
- The third side of the triangle AC is the resultant in the opposite direction.

Mathematical problem-1:

The use of the vector form of Coulomb's law consider a charge of $Q_1 = 3 \times 10^{-4}\ C$ at P (1, 2, 3) and a charge of $Q_2 = -10^{-4} C$ at Q (2, 0, 5) in a vacuum. Evaluate the force exerted on Q_2 by Q_1.

Solution:

Given,

$$Q_1 = 3 \times 10^{-4}\ C$$

$$Q_2 = -10^{-4}\ C$$

Now,

$$R_{12} = r_2 - r_1$$

$$= (2-1)a_x + (0-2)a_y + (5-3)a_z$$

$$= a_x - 2\ a_y + 2\ a_z$$

$$|R_{12}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

$$a_{12} = \frac{1}{3}\left(a_x - 2\ a_y + 2\ a_z\right)$$

Thus,

$$F_2 = \frac{Q_1\ Q_2}{R^2} a_{12}$$

$$= \frac{3\times10^{-4} \times\left(-10^{-4}\right)}{4\pi(\frac{1}{36\pi}\times10^{-9})\times3^2}\left(\frac{a_x-2\ a_y+2\ a_z}{3}\right)$$

$$= -10\left(a_x - 2\ a_y + 2\ a_z\right)$$

$$= -10a_x + 20a_y - 20a_z \quad \text{(Answer)}$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction.

$$F_1 = -F_2 = \frac{Q_1\ Q_2}{4\pi\epsilon_0\ (R_{12})^2}\ a_{21} = -\ \frac{Q_1\ Q_2}{4\pi\epsilon_0\ (R_{12})^2}\ a_{12}$$

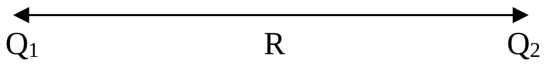
Electric Field Intensity

The electric field intensity (or electric field strength) E is the force per unit charge when placed in the electric field. Thus

$$E = \lim_{Q \to 0} \frac{F}{Q}$$

Or simply

$$E = \frac{F}{Q}$$



Now consider one charge fixed in position, say Q_1 and move a second charge slowly around, note that there exists everywhere a force on this second charges; in other words, this second charge is displaying the existence of a force field. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law,

$$F_t = \frac{Q_1\, Q_t}{R^2}\; a_{1t}$$

This force as a force per unit charge gives

$$\frac{F_t}{Q_t} = \frac{Q_1}{4\pi\epsilon_0 {R_{1t}}^2}\; a_{1t}$$

The quantity on the right side of the above equation is function only of Q_1 and the directed line segment from Q_1 to the position of the test charge. This describes a vector field and is called the electric field intensity.

We define the electric field intensity as the vector force on a unit positive test charge. Electric field intensity must be measured by the unit newtons per coulomb-the force per unit charge. We shall at once measure electric field intensity in the practical units of volts per meter (V/m). Using a capital letter **E** for electric field intensity, we have

$$E = \frac{F_t}{Q_t}$$

$$E = \frac{Q_1}{4\pi\epsilon_0 {R_{1t}}^2}\; a_{1t}$$

Finally,

$$E = \frac{Q}{4\pi\epsilon_0 R^2}\; a_R$$

Mathematical problem-2:

Point charges 1 mC and -2 mC are located at (3, 2, -1) and (-1, -1, 4) respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

$$F = \sum_{K=1,2} \frac{QQ_k}{4\pi\epsilon_0 R^2} \; a_R$$

$$= \sum_{K=1,2} \frac{QQ_k \,(r - r_k)}{4\pi\epsilon_0 \,|r - r_k|^3}$$

$$= \frac{QQ_1(r - r_1)}{4\pi\epsilon_0 |r - r_1|^3} + \frac{QQ_2(r - r_2)}{4\pi\epsilon_0 |r - r_2|^3}$$

$$= \frac{Q}{4\pi\epsilon_0} \left\{ \frac{Q_1(r - r_1)}{|r - r_1|^3} + \frac{Q_2(r - r_2)}{|r - r_2|^3} \right\}$$

$$= \frac{Q}{4\pi\epsilon_0} \left\{ \frac{10^{-3}[(0,3,1) - (3,2,-1)]}{|(0,3,1) - (3,2,-1)|^3} - \frac{2.10^{-3}\,[(0,3,1) - (-1,-1,4)]}{|(0,3,1) - (-1,-1,4)|^3} \right\}$$

$$= 9 \times 10^{-2} \left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}} \right]$$

$$F = -6.507 \; a_x - 3.817 \; a_y + 7.506 \; a_z \; mN$$

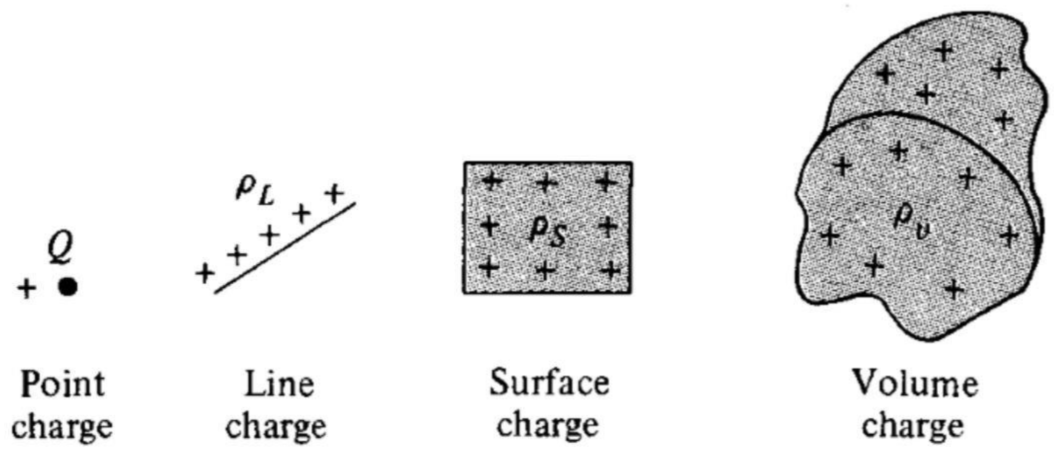
At that point,

$$E = \frac{F}{Q}$$

$$= (-6.507, -3.817, 7.506) \times \frac{10^{-3}}{10 \times 10^{-9}}$$

$$E = \; -650.7 \; a_x - 381.7 \; a_y + 750.6 \; a_z \; kV/m$$

Electric Fields due to continuous charge distributions



It is customary to denote the line charge density, surface charge density and volume charge density by ρ_L (in C/m), ρ_S (in C/m^2) and ρ_v (in (C/m^3) respectively. The charge element dQ and the total charge Q due to these charge distributions are obtained from above figure as

$$dQ = \rho_L\, dl \;\rightarrow Q = \int_L \rho_L\, dl \text{ (Line charge)}$$

$$dQ = \rho_S\, dS \;\rightarrow Q = \int_S \rho_S\, dS \text{ (Surface charge)}$$

$$dQ = \rho_v\, dv \;\rightarrow Q = \int_v \rho_v\, dv \text{ (Volume charge)}$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_S and ρ_v may be regarded as the summation of the field contributed by the numerous point charges making $\rho_L\, dl$, $\rho_S\, dS$, $\rho_v\, dv$ and integrating, we get

$$E = \int \frac{\rho_L\, dl}{4\pi\epsilon_0 R^2}\, a_R \quad \text{(Line charge)}$$

$$E = \int \frac{\rho_S\, dS}{4\pi\epsilon_0 R^2}\, a_R \quad \text{(Surface charge)}$$

$$E = \int \frac{\rho_v\, dv}{4\pi\epsilon_0 R^2}\, a_R \quad \text{(Volume charge)}$$

Field due to a continuous volume charge distribution

Cylindrical Co-ordinate

$$dQ = \rho_v dv$$

$$\int dQ = \int_v \rho_v dv$$

$$= \int_{z=0}^{h} \int_{\varphi=0}^{2\pi} \int_{0}^{r} \rho_v \, \rho d\rho \, d\varphi \, dz$$

Spherical Co-ordinate

$$dQ = \rho_v dv$$

$$\int dQ = \int_v \rho_v dv$$

$$= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{0}^{r} \rho_v \, r^2 \, sin\theta \, d\theta \, dr \, d\varphi$$

Surface of Cylindrical Co-ordinate

$$\int dQ = \int_s \rho_s ds$$

$$Q = \int_{0}^{2\pi} \int_{0}^{\pi} \rho_s \, \rho d\rho \, d\varphi \, a_z$$

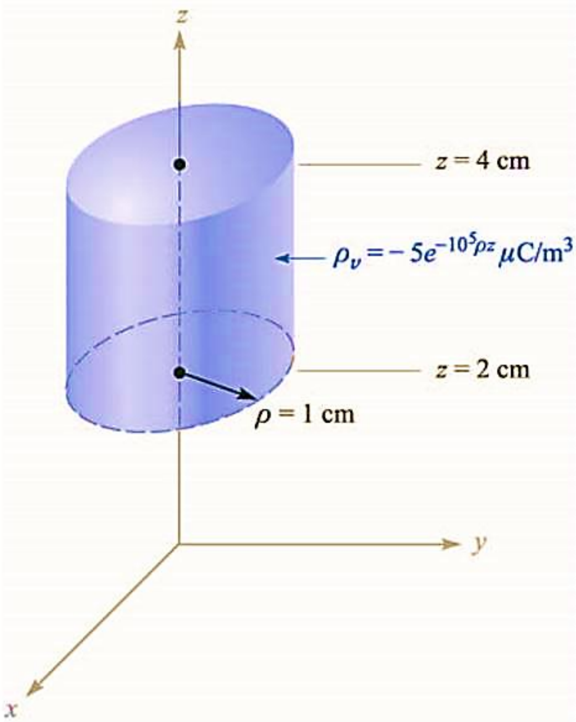
Spherical Co-ordinate:

$$\int dQ = \int_s \rho_s ds$$

$$Q = \int_{0}^{2\pi} \int_{0}^{\pi} \rho_s (\, r^2 \, sin\theta \, d\theta \, d\varphi) \, a_r$$

Mathematical problem-3:

Evaluate the total charge contained in a 2-cm length of the electron beam show in following figure.



Solution:

The total charge contained may be obtained by evaluating

$$Q = \int_v \rho_v \, dv$$

The charge density is

$$\rho_v = -5 \times 10^{-6}\, e^{-10^5}\, C/m^2$$

Therefore,

$$Q = \int_{0.02}^{0.04} \int_{0}^{2\pi} \int_{0}^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z}\, \rho d\rho\, d\varphi\, dz$$

$$= -10^{-5}\pi \int_{0.02}^{0.04} \int_{0}^{0.01} e^{-10^5 \rho z}\, \rho\, d\rho\, dz$$

$$= -10^{-5}\pi \int_{0}^{0.01} \left[\frac{e^{-10^5 \rho z}}{10^{-5}\rho}\right]_{0.02}^{0.04} \rho\, d\rho$$

$$= -10^{-10}\pi \int_{0}^{0.01} [e^{-2000\rho} - e^{-4000\rho}]\, d\rho$$

$$= -10^{-10}\pi\ (\frac{1}{2000} - \frac{1}{4000})$$

$$= -10^{-10}\pi\ (\frac{2-1}{4000})$$

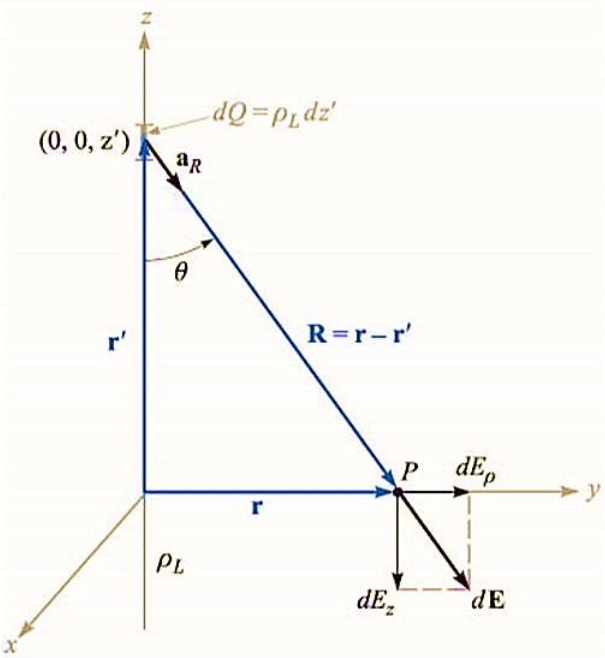
$$= -10^{-10}\pi\ (\frac{1}{40\times10^{2}})$$

$$= \frac{-\pi}{40}\ \rho c$$

= 0.0785 pC   (Ans)

Field of a Line Charge

Let us assume a straight line charge extending along the z-axis in a cylindrical co-ordinate system from $-\infty$ to ∞ as shown in following figure. We desire the electric field intensity, E at any and every point resulting from a uniform line charge density ρ_L .



As we move around the line charge, varying φ while keeping ρ and z constant, the line charge appears the same from every angle. Again, if we maintain ρ and φ constant while moving up and down the line charge by changing z, the line charge still recedes into infinite distance in both directions and the problem is unchanged. This is axial symmetry and leads to fields are not functions of z.

If we maintain φ and z constant and vary ρ, the problem changes and Coulomb's law leads us to expect the field to become weaker as ρ increases. Hence, by a process of elimination we are led to the fact that the field varies only with ρ.

No element of charge produces a φ component of electric intensity; E_φ is zero. However, each element does produce an E_ρ and an E_z component but the contribution to E_z by elements of charge which are equal distances above and below the point at which we are determining the field will cancel.

Therefore we have found that we have only an E_ρ component and it varies only with ρ. Now to find this component we choose a point P(0, y, 0) on the y-axis at which to determine the field. This is perfectly general point in view of the lack of variation of the field with φ and z. To find the incremental field at P due to the incremental charge $dQ = \rho_L dz'$, we have

$$dE = \frac{\rho_L dz' \; (r - r')}{4\pi\epsilon_0 \; |r - r'|^3}$$

Where

$$r = ya_y = \rho a_\rho$$

$$r' = z' a_z$$

And

$$r - r' = \rho a_\rho - z' a_z$$

Therefore,

$$dE = \frac{\rho_L dz' \, (\rho a_\rho - z' a_z)}{4\pi\epsilon_0 \, (\rho^2 + z'^2)^{3/2}}$$

Since only the E_ρ component is present, we may simplify:

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0 \, (\rho^2 + z'^2)^{3/2}}$$

And

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0 \, (\rho^2 + z'^2)^{3/2}}$$

Integrating by integral tables or change of variable, $z' = \rho \, cot\theta$, we have

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0} \rho \left(\frac{1}{\rho^2} \frac{z'}{\sqrt{\rho^2 + z'^2}} \right)_{-\infty}^{\infty}$$

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho} a_\rho$$

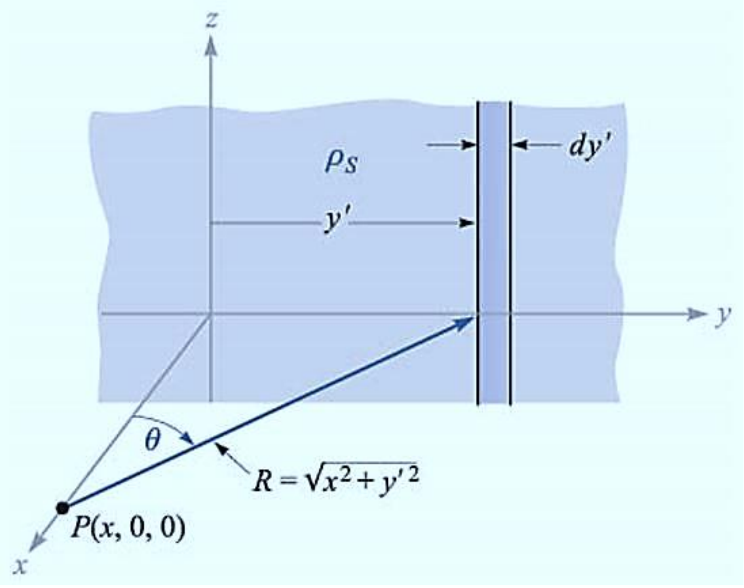
Field of a Sheet of Charge

The infinite sheet of charge having a unit form density of $\rho_s\ C/m^2$.

Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line of parallel-plate capacitor. ρ_s is commonly known as *surface charge density*.

Let us place a sheet of charge in the yz plane and again consider symmetry as following figure. The field cannot vary with y or with z and then that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we wish the field will cancel. Hence only E_x is present and this component is a function of x alone.

Let us use the field of the infinite line charge by dividing the infinite sheet into differential-width strips shown in following figure. The line charge density or charge per unit length is $\rho_L = \rho_s\, dy'$ and the distance from this line charge to out general point P on the x-axis is $R = \sqrt{x^2 + y'^2}$.



$$E = \frac{\rho_L}{2\pi\epsilon_0 R}\ a_R$$

$$dE = \frac{\rho_s dy'}{2\pi\epsilon_0 R}\ a_R$$

The contribution to E_x at P from this differential-width strip is then

$$dE_x = \frac{\rho_s dy'}{2\pi\epsilon_0\sqrt{x^2 + y'^2}}\ cos\theta = \frac{\rho_s}{2\pi\epsilon_0}\ \frac{x dy'}{x^2 + y'^2}$$

Adding the effects of all the strips,

$$E_x = \frac{\rho_s}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x dy'}{x^2 + y'^2}$$

$$= \frac{\rho_s}{2\pi\epsilon_0}\ \tan^{-1}\frac{y'}{x}\Big]_{-\infty}^{-}$$

$$= \frac{\rho_s}{2\epsilon_0}$$

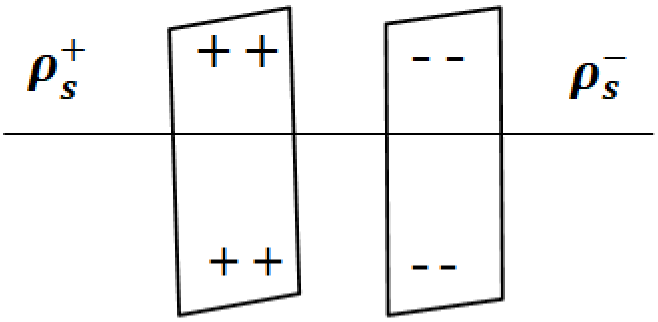
If the point P were chosen on the negative x-axis, then

$$E_x = -\frac{\rho_s}{2\epsilon_0}$$

For the field is always directed away from the positive charge. This difficulty in sign in usually overcome by specifying a unit vector a_N, which is normal to the sheet and directed outward or away from it. Then

$$E = \frac{\rho_s}{2\epsilon_0}\, a_N$$

If a second infinite sheet of charge having a negative charge density $-\rho_s$, is located in the plane $x = a$, we may find the total field by adding the contribution of each sheet.



In the region $x > a$,

$$E_+ = \frac{\rho_s}{2\epsilon_0}\, a_x$$

$$E_- = -\frac{\rho_s}{2\epsilon_0}\, a_x$$

$$E = E_+ + E_- = 0$$

And for $x < 0$,

$$E_+ = -\frac{\rho_s}{2\epsilon_0}\, a_x$$

$$E_- = \frac{\rho_s}{2\epsilon_0}\, a_x$$

$$E = E_+ + E_- = 0$$

And when $0 < x < a$,

$$E_+ = \frac{\rho_s}{2\epsilon_0}\, a_x$$

$$E_- = \frac{\rho_s}{2\epsilon_0}\, a_x$$

$$E = E_+ + E_- = \frac{\rho_s}{\epsilon_0}\, a_x$$