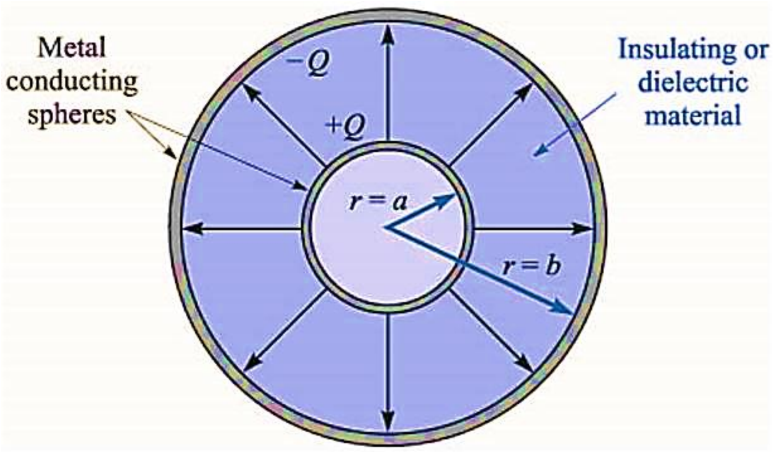
CHAPTER-3

Electric Flux Density, Gauss's Law and Divergence

Electric Flux Density

Electric flux density is electric flux passing through a unit area perpendicular to the direction of the flux. Electric flux density is a measure of the strength of an electric field generated by a free electric charge, corresponding to the number of electric lines of force passing through a given area.



A larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere. If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\Psi = Q$$

The electric flux Ψ is measured in coulombs.

Considering-

- Inner sphere radius a
- Outer sphere radius b
- With charges Q and $-Q$ respectively.

Then,

Surface area of the sphere, $S = 4\pi r^2$

Flux produced per unit area = $\frac{\Psi}{S}$

The electric flux density,

$$D|_{r=a} = \frac{Q}{4\pi a^2} a_r \quad \text{(Inner sphere)}$$

$$D|_{r=b} = \frac{Q}{4\pi b^2} a_r \quad \text{(Outer sphere)}$$

At a radial distance r, where $a \leq r \leq b$,

The electric flux density, $D = \frac{Q}{4\pi r^2} a_r$

And the Electric field intensity for a point charge, $E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$

In free space, therefore, the relationship between electric flux density (D) and electric field intensity (E),

$$E = \frac{D}{\epsilon_0} \quad \text{(Free space only)}$$

$$D = \epsilon_0 E \quad \text{(free space only)}$$

Mathematical problem-1

Evaluate electric flux density, D in the region about a uniform line charge of 8 nC/m lying along the z-axis in free space.

Solution:

Given that,

$$\rho_L = 8\, nC/m = 8 \times 10^{-9}\ C/m$$

And, $\rho = 3m$

We know, the field of a line charge,

$$E = \frac{\rho_L}{2\pi\epsilon_0\rho} a_\rho$$

$$= \frac{8\times10^{-9}}{2\times(3.1416)\times(8.854\times10^{-12})\times3} a_\rho$$

$$= 47.93\ a_\rho \text{ V/m.}$$

We know,

$$D = \epsilon_0 E$$

$$= (8.854 \times 10^{-12}) \times 47.93$$

$$= 4.24 \times 10^{-10}\ \frac{C}{m}$$

$$= 0.424\ nC/m$$

If, $\rho_L = 8\ nC$ and the total flux leaving a (5-m) length of the line charge then

Given that,

$$L = 5m$$

and $\rho_L = 8\ nC = 8 \times 10^{-9}\ C$

We know,

$$dQ = \rho_L d_L$$

$$\Rightarrow Q = (8 \times 10^{-9}) \times 5$$

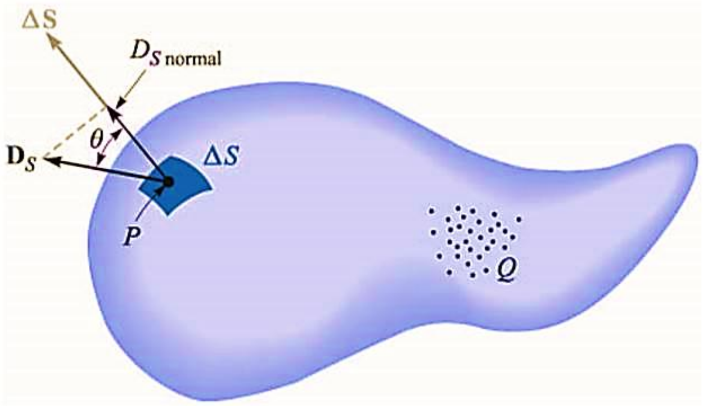
$$\therefore Q = 40\ nC$$

Gauss's Law

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

Thus

$$\Psi = Q_{enclosed}$$



The electric flux density D_S at P due to charge Q. The total flux passing through ΔS is $D_S \cdot \Delta S$.

$$\text{Flux crossing } \Delta S = \Delta\Psi = D_S S\, cos\theta \Delta S = D_S \cdot \Delta S$$

$$\Delta\Psi = D_S \cdot \Delta S$$

The total flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element ΔS,

$$\Psi = \int d\Psi = \oint_{\substack{closed \\ surface}} D_S \cdot dS$$

Then the mathematical formulation of Gauss's law,

$$\Psi = \oint_S D_S \cdot dS = charge\ enclosed = Q$$

The charge enclosed might be several point charges, in which case

$$Q = \sum Q_n$$

Or a line charge,

$$Q = \int \rho_L dL$$

Or a surface charge,

$$Q = \int_S \rho_S \, dS \quad (Not\ necessarity\ a\ closed\ surface)$$

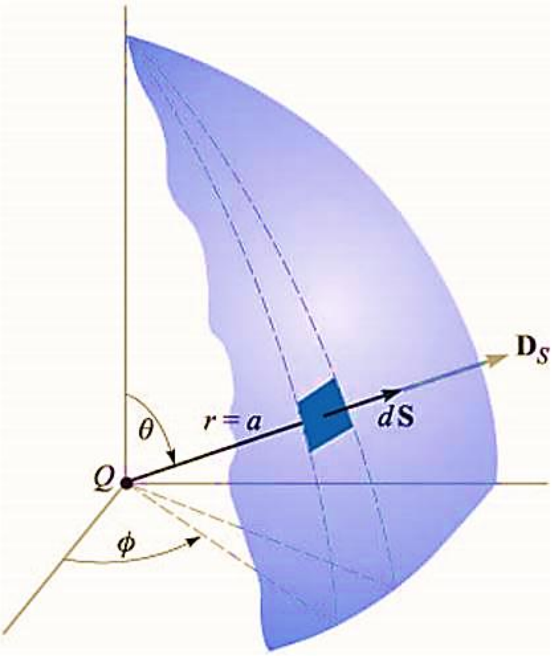
Or a volume charge,

$$Q = \int_{vol} \rho_v dv$$

Gauss's law may be written in terms of the charge distribution as a mathematical statement meaning simply that the total electric flux through any closed surface is equal to the charge enclosed.

$$\oint_S D_S \cdot dS = \int_{vol} \rho_v dv$$

Using Gauss's Law obtain electric field intensity, E and electric flux density, D for a point charge on a spherical closed surface.



At the surface of a sphere,

$$D_S = \frac{Q}{4\pi r^2}\ a_r$$

The differential element of area on a spherical surface is,

$$dS = \ r^2 \ sin\theta\ d\theta\ d\varphi\ a_r$$

The integrand is,

$$D_S . dS = \frac{Q}{4\pi r^2}\ r^2\ sin\theta\ d\theta\ d\varphi\ \ a_r \cdot\ a_r$$

$$= \frac{Q}{4\pi}\ \ sin\theta\ d\theta\ d\varphi$$

Then,

$$Q = \oint D_s . dS$$

$$= \oint D_s . r^2\ sin\theta\ d\theta\ d\varphi$$

$$= D_s r^2 \int_0^{2\pi} \int_0^{\pi} sin\theta\ d\theta\ d\varphi$$

$$= D_s r^2\ [-cos]_0^{\pi}\ [\varphi]_0^{2\pi}$$

$$= D_s r^2\ [1 + 1]\ [2\pi - 0]$$

$$= D_s r^2\ 2 \cdot 2\pi$$

$$= 4\pi D_s r^2$$

Here $D_s = \frac{Q}{4\pi r^2}$

Electric flux density, D for a point charge,

$$D = \frac{Q}{4\pi r^2}\ a_r$$

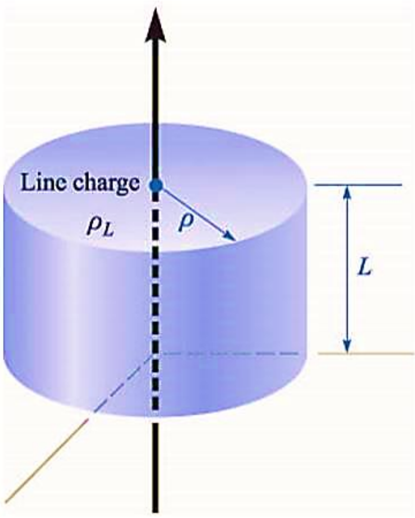
Electric field intensity, E for a point charge,

We know,

$$D = \epsilon_0 E$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$$

Using Gauss's Law obtain electric field intensity, E and electric flux density, D for a line charge.



We know,

Differential surface of cylinder,

$$ds = \rho d\varphi\ dz\ a_\rho$$

Then,

$$Q = \oint \mathrm{D}_s . dS$$

$$= \oint \mathrm{D}_s . \rho d\varphi\ dz\ a_\rho$$

$$= \mathrm{D}_s \int_0^l \int_0^{2\pi} \rho d\varphi\ dz$$

$$= \mathrm{D}_s \rho\ [y]_0^{2\pi}\ [z]_0^l$$

$$= \mathrm{D}_s\ \rho\ 2\pi\ l$$

Here $\mathrm{D}_s = \frac{\mathrm{Q}}{2\pi\rho l}$

Electric flux density, D for a line charge,

$$D = \frac{\rho l}{2\pi\rho}\ a_\rho$$

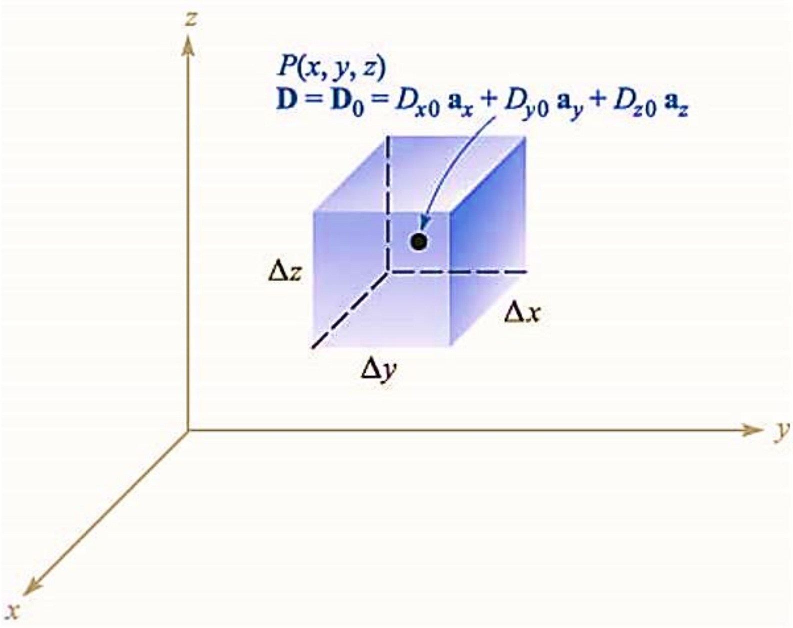
Electric field intensity, E for a line charge,

We know,

$$D = \epsilon_0 E$$

$$E = \frac{\rho l}{2\pi\epsilon_0\rho}\ a_\rho$$

Application of Gauss's Law: Differential Volume Element



Consider point P, shown in figure located by a Cartesian coordinate system. The value of D at the point P may be expressed in Cartesian components, $D_0 = D_{x0}\, a_x + D_{y0}\, a_y + D_{z0}\, a_z$. We chose as our closed surface the small rectangular box, centered at P, having sides of lengths $\Delta x, \Delta y\ and\ \Delta z$.

Applying Gauss's law,

$$Q = \oint_{Surface} D_s \cdot dS$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S D_s \cdot dS = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom} \quad \text{\_\_\_\_\_\_\_\_\_\_} (1)$$

Consider,

$$\int_{front} = D_{front} \cdot \Delta S_{front}$$

$$= D_{front} \cdot \Delta y\, \Delta z\, a_x$$

$$= D_{x,front}\, \Delta y\, \Delta z$$

D_{x0} is the value of D_x, the front surface is at a distance of $\Delta x/2$ from P and partial derivative must be used to express the rate of change of D_x with x. Then,

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \times rate\ of\ change\ of\ D_x\ with\ x$$

$$= D_{x0} + \frac{\Delta x}{2}\frac{\delta D_x}{\delta x}$$

Now we have,

$$\int_{front} = \left(D_{x0} + \frac{\Delta x}{2}\frac{\delta D_x}{\delta x}\right) \Delta y \, \Delta z$$

And the back surface,

$$\int_{back} = \left(-D_{x0} + \frac{\Delta x}{2}\frac{\delta D_x}{\delta x}\right) \Delta y \, \Delta z$$

Now,

$$\int_{front} + \int_{back} = D_{x0} \cdot \Delta y \, \Delta z + \frac{\Delta x}{2}\frac{\delta D_x}{\delta x} \cdot \Delta y \, \Delta z - D_{x0} \cdot \Delta y \, \Delta z + \frac{\Delta x}{2}\frac{\delta D_x}{\delta x} \cdot \Delta y \, \Delta z$$

$$= \frac{\delta D_x}{\delta x}\left(\frac{\Delta x}{2} + \frac{\Delta x}{2}\right) \Delta y \, \Delta z$$

$$= \frac{\delta D_x}{\delta x} \, \Delta x \, \Delta y \, \Delta z$$

Similarly,

$$\int_{right} + \int_{left} = \frac{\delta D_y}{\delta y} \, \Delta x \, \Delta y \, \Delta z$$

And

$$\int_{top} + \int_{bottom} = \frac{\delta D_z}{\delta z} \, \Delta x \, \Delta y \, \Delta z$$

Now from equation (1), we have,

$$\oint_S D_s \cdot \mathrm{dS} = \left(\frac{\delta D_x}{\delta x} \, \Delta x \, \Delta y \, \Delta z\right) + \left(\frac{\delta D_y}{\delta y} \, \Delta x \, \Delta y \, \Delta z\right) + \left(\frac{\delta D_z}{\delta z} \, \Delta x \, \Delta y \, \Delta z\right)$$

$$\oint_S D_s \cdot \mathrm{dS} = \left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}\right) \Delta x \, \Delta y \, \Delta z$$

$$\oint_S D_s \cdot \mathrm{dS} = \left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}\right) \Delta v$$

We can write,

$$Q = \oint_S D_s \cdot \mathrm{dS} = \left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}\right) \Delta v$$

$$\frac{Q}{\Delta v} = \frac{\oint_S D_s \cdot \mathrm{dS}}{\Delta v} = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

Charge enclosed in volume $\Delta v = \left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}\right) \times volume \Delta v$

Divergence

$$\frac{Q}{\Delta v} = \frac{\oint_S D_s \cdot \mathrm{dS}}{\Delta v} = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

By allowing the volume element Δv shrink to zero, we can write

$$\left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S D_s \cdot \mathrm{dS}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

$$\left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S D_s \cdot \mathrm{dS}}{\Delta v} = \rho_v$$

The equation on any vector D to find $\oint_S \mathrm{D} \cdot \mathrm{dS}$ for a small closed surface,

$$\left(\frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \mathrm{D} \cdot \mathrm{dS}}{\Delta v}$$

The divergence of the vector flux density D is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

$$Divergence\ of\ D = div\ D = \lim_{\Delta v \to 0} \frac{\oint_S \mathrm{D} \cdot \mathrm{dS}}{\Delta v}$$

Then,

$$div\ D = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z} \quad (Cartesian)$$

$$div\ D = \frac{1}{\rho}\frac{\delta}{\delta \rho}\left(\rho D_\rho\right) + \frac{1}{\rho}\frac{\delta D_\varphi}{\delta \varphi} + \frac{\delta D_z}{\delta z} \quad (Cylindrical)$$

$$div\ D = \frac{1}{r^2}\frac{\delta}{\delta r}(r^2 D_r) + \frac{1}{r\ sin\theta}\frac{\delta}{\delta \theta}(sin\theta\ D_\theta) + \frac{1}{r\ sin\theta}\frac{\delta D_\varphi}{\delta \varphi} \quad (Spherical)$$

Maxwell's First equation (Electrostatics)

$$div\ D = \lim_{\Delta v \to 0} \frac{\oint_S \mathrm{D} \cdot \mathrm{dS}}{\Delta v}$$

$$div\ D = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

Then,

$$\oint_S \mathrm{D} \cdot \mathrm{dS} = \mathrm{Q}$$

Per unit volume

$$\frac{\oint_S \mathrm{D} \cdot \mathrm{dS}}{\Delta v} = \frac{Q}{\Delta v}$$

As the volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathrm{D} \cdot \mathrm{dS}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

We can write,

$$div\ D = \rho_v$$

Integral Form of Maxwell's equation,

$$\oint_S \mathrm{D} \cdot \mathrm{dS} = \int_v \rho_v\ dv = Q$$

Differential Form of Maxwell's equation,

$$\nabla \cdot D = \rho_v$$

Mathematical problem-2

Find an approximate value for the total charge enclosed in an incremental volume of $10^{-9} m^3$ located at the origin, if $D = e^{-x} \sin y\, a_x - e^{-x} \cos y\, a_y + 2z\, a_z \;\; C/m^2$.

Solution:

Evaluate the three partial derivatives,

$$\frac{\delta D_x}{\delta x} = -e^{-x}\, sin\, y$$

$$\frac{\delta D_y}{\delta y} = e^{-x}\, sin\, y$$

$$\frac{\delta D_z}{\delta z} = 2$$

- At the origin, the first two expressions are zero and the last is 2.
- Thus the charge enclosed in a small volume element there must be approximately $2\Delta v$.

If $\Delta v = 10^{-9}\, m^3$, then we have the total charge enclosed about 2 nC. (Ans)

Mathematical problem-3

Find div D at the origin if $D = e^{-x}\, sin\, y\, a_x - e^{-x}\, cos\, y\, a_y + 2z\, a_z$

Solution:

We know

$$div\ D = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

$$= -e^{-x} \sin y + e^{-x} \sin y + 2$$

$$= 2 \quad (Ans)$$

Mathematical problem-4

Determine the divergence of the vector field $D = x^2yz\, a_x + xz\, a_z$

Solution:

We know

$$div\ D = \frac{\delta D_x}{\delta x} + \frac{\delta D_y}{\delta y} + \frac{\delta D_z}{\delta z}$$

$$= \frac{\delta}{\delta x}\ (x^2yz\,) + \frac{\delta}{\delta y}\ (0) + \frac{\delta}{\delta z}\ (xz)$$

$$= 2xyz + x \quad (Ans)$$

Mathematical problem-5

Determine the divergence of the vector field $D = \rho\ sin\varphi\ a_\rho + \rho^2 z\ a_\varphi + z\ cos\varphi\ a_z$

Solution:

We know

$$div\ D = \frac{1}{\rho}\frac{\delta}{\delta \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\delta}{\delta \varphi}D_\varphi + \frac{\delta}{\delta z}D_z$$

$$= \frac{1}{\rho}\frac{\delta}{\delta \rho}(\rho^2\ sin\varphi) + \frac{1}{\rho}\frac{\delta}{\delta \varphi}(\rho^2 z) + \frac{\delta}{\delta z}\ (z\ cos\varphi)$$

$$= 2\ sin\varphi + cos\varphi \quad (Ans)$$

Mathematical problem-6

Determine the divergence of the vector field $D = \frac{1}{r^2}\ cos\theta\ a_r + r\ sin\theta\ cos\varphi\ a_\theta + cos\theta\ a_\varphi$

Solution:

We know

$$div\ A = \frac{1}{r^2}\frac{\delta}{\delta r}(r^2 D_r) + \frac{1}{r\ sin\theta}\frac{\delta}{\delta \theta}(sin\theta\ D_\theta) + \frac{1}{r\ sin\theta}\frac{\delta}{\delta \varphi}D_\varphi$$

$$= \frac{1}{r^2}\frac{\delta}{\delta r}\ (cos\theta) + \frac{1}{r\ sin\theta}\frac{\delta}{\delta \theta}\ (r\ sin^2\theta\ cos\varphi) + \frac{1}{r\ sin\theta}\frac{\delta}{\delta \varphi}\ (cos\theta)$$

$$= 0 + \frac{1}{r\ sin\theta}\ 2r\ sin\theta\ cos\theta\ cos\varphi + 0$$

$$= 2\ cos\theta\ cos\varphi \quad (Ans)$$

Mathematical problem-7

Evaluate both sides of the divergence theorem for the field $D = 2xy\, a_x + x^2\, a_y\ C/m^2$ and the rectangular parallelepiped formed by the planes $x = 0\ and\ 1, y = 0\ and\ 2, and\ z = 0\ and\ 3$.

Solution:

Given

$$D = 2xy\, a_x + x^2\, a_y$$

$$x = 0\ and\ 1$$

$$y = 0\ and\ 2$$

$$z = 0\ and\ 3$$

We know,

$$\int_{vol} (\nabla \cdot D)\, dv = \oint_S D \cdot ds$$

Then,

$$\nabla . D = \frac{\delta}{\delta x}\left(2xy\, a_x + x^2 a_{\ y}\right) a_x + \frac{\delta}{\delta y}\left(2xy\, a_x + x^2\, a_y\right) a_y + \frac{\delta}{\delta z}\left(2xy\, a_x + x^2\, a_y\right) a_z$$

$$= \frac{\delta}{\delta x}\ 2xy + \frac{\delta}{\delta y}\ x^2 + \frac{\delta}{\delta z}$$

$$= 2y$$

Now,

$$L.H.S = \int_{vol} (\nabla \cdot D)\, dv$$

$$= \int_{z=0}^{3} \int_{y=0}^{2} \int_{x=0}^{1} (2y)\, dx\, dy\, dz$$

$$= [x]_0^1\, [y]_0^2\, [z]_0^3$$

$$= 1 \times 2 \times 3$$

$$= 12\ C$$

Again,

$$R.H.S = \oint_S D \cdot ds$$

$$= \int_{z=0}^{3} \int_{y=0}^{2} (D)_{x=0} \cdot (-dy\, dz\, a_x) + \int_{z=0}^{3} \int_{y=0}^{2} (D)_{x=1} \cdot (dy\, dz\, a_x)$$
$$+ \int_{z=0}^{3} \int_{y=0}^{2} (D)_{y=0} \cdot \left(-dx\, dz\, a_y\right) + \int_{z=0}^{3} \int_{y=0}^{2} (D)_{y=2} \cdot \left(dx\, dz\, a_y\right)$$

$$= -\int_0^3 \int_0^2 (D)_{x=0} \cdot dy\, dz + \int_0^3 \int_0^2 (D)_{x=1} \cdot dy\, dz - \int_0^3 \int_0^2 (D)_{y=0} \cdot dx\, dz + \int_0^3 \int_0^2 (D)_{y=2} \cdot dx\, dz$$

However, $(D)_{x=0} = 0 \; and \; (D)_{y=0} = (D)_{y=2}$, which leaves only

$$= \int_0^3 \int_0^2 (D)_{x=1} \cdot dy\, dz$$

$$= \int_0^3 \int_0^2 2y \cdot dy\, dz$$

$$= \int_0^3 4\, dz$$

$$= 12$$

A total charge of 12 C lies within this parallelepiped. (Ans)

Mathematical problem-8

Given the flux density, $D = \frac{16}{r} cos2\theta a_\theta \ C/m^2$, use two different methods to find the total charge within the region $1 < r < 2m, 1 < \theta < 2rad, 1 < \varphi < 2rad.$

Solution:

Evaluating the net outer flux through a cube,

Here, D has only θ component so the flux contributions will be only through the surface of constant θ. On a constant–theta surface, the differential area is $da = r\ Sin\theta\ dr\ d\varphi$, where θ is fixed at surface location.

Flux integral- (For the both surface of θ)

$$\oint D\,.ds = \int_{Left} D\,.ds + \int_{Right} D\,.ds + \int_{Front} D\,.ds + \int_{Back} D\,.ds + \int_{Top} D\,.ds + \int_{Bottom} D\,.ds$$

$$= 0 + 0 + 0 + 0 - \int_1^2 \int_1^2 \frac{16}{r} \cos(2)\ r\ Sin\ (1) dr\ d\varphi + \int_1^2 \int_1^2 \frac{16}{r} \cos(4)\ r\ Sin\ (2) dr\ d\varphi$$

$$= -16 \int_1^2 \int_1^2 \cos(2) Sin\ (1) dr\ d\varphi + 16 \int_1^2 \int_1^2 \cos(4) Sin\ (2) dr\ d\varphi$$

$$= -16 \times -0.3502 \times [r]_1^2 \times [\varphi]_1^2 + 16 \times -0.5943 \times [r]_1^2 \times [\varphi]_1^2$$

$$= 16 \times 0.3502 - 16 \times 0.5943$$

$$= 16\ (0.3502 - 0.5943)$$

$$= -3.91\ C \quad (Ans)$$

The volume integral side of the divergence theorem,

$$\nabla\,.D = \frac{1}{r^2} \frac{d}{dr} (r^2\ D_r) + \frac{1}{r\ Sin\ \theta} \frac{d}{d\theta} (Sin\ \theta\ D_\theta) + \frac{1}{r\ Sin\ \theta} \frac{d}{d\varphi} (D\varphi)$$

$$= 0 + \frac{1}{r\ Sin\ \theta} \frac{d}{d\theta} (Sin\ \theta\ D_\theta) + 0$$

$$= \frac{1}{r\ Sin\theta} \frac{d}{d\theta} \left[\frac{16}{r} \cos 2\theta\ Sin\theta\right]$$

$$= \frac{16}{r^2} \left[\frac{\cos 2\theta \cos\theta}{Sin\ \theta} - 2\ Sin\ 2\theta\right]$$

Now,

$$dv = r^2\ Sin\theta\ dr\ d\theta\ d\varphi$$

Then,

$$\int_{volume} \nabla . D\, dv = \int_1^2 \int_1^2 \int_1^2 \frac{16}{r^2} \left[\frac{\cos 2\theta\ cos\theta}{Sin\ \theta} - \ 2\ Sin\ 2\theta\right] r^2\ Sin\ \theta\ dr\ d\theta\ d\varphi$$

$$= \int_1^2 \int_1^2 \int_1^2 16 \left[\frac{\cos 2\theta \cos\theta - 2\ Sin\ 2\theta\ Sin\ \theta}{Sin\ \theta}\right] Sin\ \theta\ dr\ d\theta\ d\varphi$$

$$= \int_1^2 \int_1^2 \int_1^2 16\ [cos\ 2\theta \cos\theta - 2\ Sin\ 2\theta\ Sin\ \theta\ ]\ \ dr\ d\theta\ d\varphi$$

$$= \int_1^2 \int_1^2 16\ [\cos 2\theta \cos\theta - 2\ Sin\ 2\theta\ Sin\ \theta]\ \ [r]_1^2\ d\theta\ d\varphi$$

$$= \int_1^2 16\ [\cos 2\theta \cos\theta - 2\ Sin\ 2\theta\ Sin\ \theta]\ [\varphi]_1^2\ d\theta$$

$$= \int_1^2 16\ [\cos 2\theta\ cos\ \theta - 2\ Sin\ 2\theta\ Sin\ \theta]\ d\theta$$

$$= 8 \int_1^2 [2\cos 2\theta \cos\theta - 4\ Sin\ 2\theta\ Sin\ \theta]\ d\theta$$

$= 8\ \int_1^2 [\cos 3\theta + \cos\theta - \ 2\ (2\ Sin\ 2\theta\ Sin\theta)]$ $\quad [2\ cosA\ cosB = \cos(A+B) + \cos(A-B)]$

$= 8\ \int_1^2 cos3\theta + \cos\theta - 2\ (cos\theta - \cos 3\theta)$ $\quad [2\ SinA\ Sin\ B = \cos(A-B) - \cos(A+B)]$

$$= 8 \int_1^2 cos3\theta + \cos\theta - 2\ cos\theta + 2\cos 3\theta$$

$$= 8 \int_1^2 [3\cos 3\theta - \cos\theta]\ \ d\theta$$

$$= 8 \left[3\ \frac{Sin\ 3\theta}{3} - \ Sin\ \theta\right]_1^2$$

$$= \ [Sin\ 6 - Sin\ 2 - Sin\ 3 + Sin\ 1]$$

$$= \ -3.91\ C \quad (Ans)$$

Mathematical problem-9

The cylindrical surface $\rho = 8\ cm$ contains the surface charge density, $\rho_s = 5\ e^{-20\,|z|}\ n\,C/m^2$.

(a) What is the total amount of charge present?

(b) How much electric flux leaves the surface, $= 8\ cm, 1\ cm < z < 5\ cm, 30° < \varphi < 90°$?

Solution:

(a) Integrate over the surface-

$$Q = \int_S \rho_s\, ds$$

$$= \int_0^{\infty}\int_0^{2\pi} \rho_s\, \rho\, d\varphi\, dz$$

$$= \int_0^{\infty}\int_0^{2\pi} 5e^{-20z}(0.08) d\varphi\, dz$$

$$= \int_0^{\infty} 5\ e^{-20\,z}\ (0.08) \int_0^{2\pi} 1\ d\varphi\, dz$$

$$= \int_0^{\infty} 5\ e^{-20\,z}\ (0.08)[\varphi]_0^{2\pi}\ dz$$

$$= 10\pi\ (0.08) \int_0^{\infty} e^{-20\,z}\ dz$$

$$= 10\pi\ (0.08) \left[\frac{e^{-20\,z}}{-20}\right]_0^{\infty}$$

$$= 10\pi\ (0.08) \times \frac{1}{-20}\ [e^{-20\,z}]_0^{\infty}$$

$$= 10\pi\ (0.08) \times \frac{1}{-20}\ (0-1)$$

$$= 10\pi\ \times\ (0.08)\ \times \frac{1}{20}$$

$$= 0.125\ nC\ (Nano - Coulomb) \quad (Ans)$$

(b) Integrate the charge density on that surface-

$$\varphi = Q' = \int_s \rho_s \, ds$$

$$= \int_{0.01}^{0.05} \int_{30°}^{90°} \rho_s \, \rho \, d\varphi \, dz$$

$$= \int_{0.01}^{0.05} \int_{30°}^{90°} 5\, e^{-20\,z} \,(0.08) d\varphi \, dz$$

$$= \int_{0.01}^{0.05} 5\, e^{-\,20\,z} \,(0.08) \int_{30°}^{90°} d\varphi \, dz$$

$$= \int_{0.01}^{0.05} 5\, e^{-\,20\,z} \,(0.08)[\varphi]_{30°}^{90°} \, dz$$

$$= \int_{0.01}^{0.05} 5\, e^{-\,20\,z} \,(0.08) \times \frac{(90-30)2\pi}{360} \, dz$$

$$= 5 \times (0.08) \times \left(\frac{90-30}{360}\right) 2\pi \int_{0.01}^{0.05} e^{-\,20\,z} \, dz$$

$$= \frac{300 \times 0.08 \times 2\pi}{360} \left[\frac{1}{-\,20}\, e^{-\,20\,z}\right]_{0.01}^{0.05}$$

$$= \frac{24 \times 2\pi}{360} \left[\frac{1}{-\,20} \left(e^{-\,20 \times 0.05} - e^{-\,20 \times 0.01}\right)\right]$$

$$= 9.45 \times 10^{-3} \, nC$$

$$= 9.45 \, pC \; (Pico - Coulomb) \quad (Ans)$$

Mathematical problem-10

Volume charge density is located in free space as $\rho_v = 2e^{-1000r}\ nC/m^3$ for $0 < r < 1\ mm$ and $\rho_v = 0$ elsewhere.

(a) Find the total charge enclosed by the spherical surface $r = 1\ mm$.

(b) By using Gauss's law, calculate the value of D_r on the surface $r = 1mm$.

Solution:

(a) The charge-

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{0.001} 2\ e^{-1000\ r}\ r^2\ Sin\ \theta\ dr\ d\theta\ d\varphi$$

We obtain,

$$Q = 8\pi \left[\left. \frac{-\ r^2\ e^{-1000\ r}}{1000} \right|_0^{0.001} + \left. \frac{2}{1000}\ \frac{e^{-1000\ r}}{(1000)^2}\ (-1000r - 1) \right|_0^{0.001} \right]$$

$$= 4.0\ \times 10^{-9}\ nC \quad (Ans)$$

(b) The enclosed charge is the result of part-(a).

We thus write $4\pi r^2 D_r = Q$.

Then,

$$D_r = \frac{Q}{4\pi r^2}$$

$$= \frac{4.0\ \times\ 10^{-9}}{4\pi\ (0.001)^2}$$

$$= 3.2\ \times\ 10^{-4}\ \frac{nC}{m^2} \quad (Ans)$$