CHAPTER-4

Energy and potential

Energy expended in moving a Point charge in an Electric Field

We wish to move a charge Q a distance dL from (B) to (A) location in an electric field E. The force on Q due to the electric field is

$$F_E = QE$$

The force which we must apply is equal and opposite to the force due to the field. Therefore, the work done is,

$$F \cdot dL = -QE.dL\, a_L$$

Where a_L= a unit vector in the direction of dL.

The differential work done by external source moving Q,

$$= -QE.dL\, a_L$$

$$= -QE.dL$$

$$\therefore dW = -QE.dL$$

- ***If E and L are perpendicular, the differential work will be zero.***

The total work required to move the charge from (B) to (A) location is,

$$W = dW$$

$$\Rightarrow W = \int_B^A - QE.dL$$

$$\therefore W = -Q \int_B^A E.dL$$

- ***W > 0 means we expend energy or do work.***
- ***W < 0 means the field expends energy or do work.***

Mathematical problem-1:

The non-uniform field $E = y\, a_x + x\, a_y + 2\, a_z$.

(a) Determine the work expended in carrying 2 C from B(1, 0, 1) to A(0.8, 0.6, 1) along the shorter arc of the circle $x^2 + y^2 = 1\ and\ z = 1$.

(b) Determine the work required to carry 2 C from B to A in the same field, but this time use straight-line path from B to A.

Solution:

(a) Working in Cartesian co-ordinates, the differential path $\boldsymbol{dL\ is\ dx\ a_x + dy\ a_y + dz\ a_z}$ and the integral becomes

$$W = -Q \int_B^A E \cdot dL$$

$$= -2 \int_B^A (y\, a_x + x\, a_y + 2\, a_z) \cdot (dx\, a_x + dy\, a_y + dz\, a_z)$$

$$= -2 \int_1^{0.8} y\, dx - 2 \int_0^{0.6} x\, dy - 4 \int_1^1 dz$$

$$= -2 \int_1^{0.8} \sqrt{1 - x^2}\, dx - 2 \int_0^{0.6} \sqrt{1 - y^2}\, dy - 0$$

$\left[\boldsymbol{Circle\ equation: x^2 + y^2 = 1, x = \sqrt{1 - y^2}, y = \sqrt{1 - x^2}}\right]$

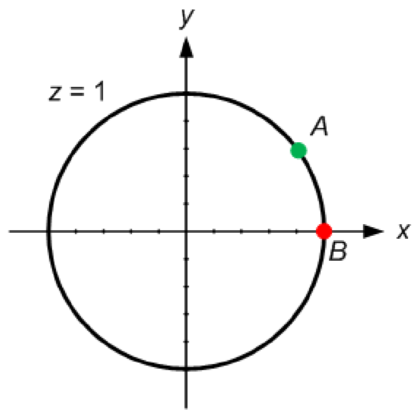
$$= -\left[x\sqrt{1 - x^2} + \sin^{-1} x\right]_1^{0.8} - \left[y\sqrt{1 - y^2} + \sin^{-1} y\right]_0^{0.6}$$

$\left[\boldsymbol{\int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a}}\right]$

$$= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0)$$

$$= -0.96\, J \quad (Ans)$$

(b) The equations of the straight line,



$$y - y_B = \frac{y_A - y_B}{x_A - x_B}\,(x - x_B)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}\,(y - y_B)$$

$$x - x_B = \frac{x_A - x_B}{z_A - z_B}\,(z - z_B)$$

From the first equation above we have

$$y = -3x + 3$$

$$y = -3\,(x - 1)$$

And from the second we obtain

$$z = 1$$

Thus,

$$W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^{1} dz$$

$$= 6\int_1^{0.8}(x-1)\,dx - 2\int_0^{0.6}\left(1 - \frac{y}{3}\right)dy$$

$$= -0.96\,J \quad (Ans)$$

Differential Length

$$d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$$ **(Rectangular)**

$$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z$$ **(Cylindrical)**

$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$$ **(Spherical)**

Electric potential

We know, the total work required to move the charge from (B) to (A) location is,

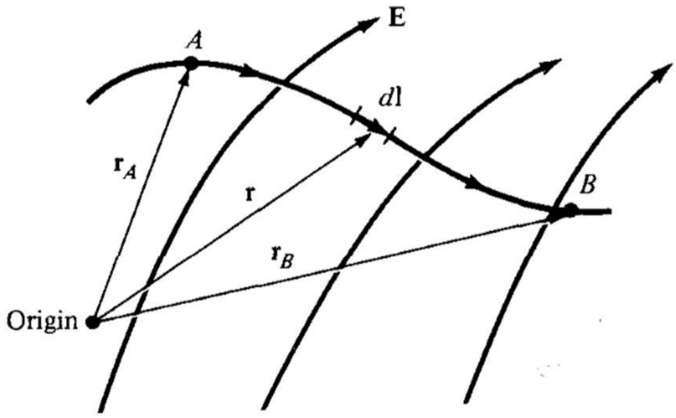
$$W = -Q\int_B^A E.dL$$

Now dividing W by Q gives the potential energy per unit charge. The quantity denoted by V_{AB} is know as the potential difference between points B and A. Thus,

$$V_{AB} = \frac{W}{Q} = -\int_B^A E.dL$$

Note that,

- In determining V_{AB}, B is the initial point while A is the final point.
- If V_{AB} is negative, there is a loss in potential energy in moving Q from B to A; this implies that the work is being done by the field. However, if V_{AB} is positive, there is a gain in potential energy in the movement.
- V_{AB} is measured in joules per coulomb, commonly referred to as volts (V).



If the E field in above figure is due to a point charge Q located at the origin, then

$$E = \frac{Q}{4\pi\epsilon_0 r^2}\ a_r$$

Then we have,

$$V_{AB} = -\int_B^A E.dL$$

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\ a_r \cdot dr\, a_r$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Then,

$$V_{AB} = V_A - V_B$$

Where, $V_B\ and\ V_A$ are the potentials at B and A respectively.

Potential difference produced by a line charge

We know,

$$E = E_\rho a_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}\ a_\rho$$

The potential difference,

$$V_{AB} = -\int_B^A E \cdot dL$$

$$= \int_B^A \left(\frac{\rho_L}{2\pi\epsilon_0\rho}\ a_\rho\right)\left(d\rho\ a_\rho + \rho d\varphi\ a_\varphi + dz\ a_z\right)$$

$$= -\frac{\rho_L}{2\pi\epsilon_0}\int_B^A \frac{d\rho}{\rho}$$

$$= -\frac{\rho_L}{2\pi\epsilon_0}\ [\ln\rho]_B^A$$

$$V_{AB} = -\frac{\rho_L}{2\pi\epsilon_0}\ \ln\frac{A}{B}$$

Work done when displacement (ρ) of source to point charge increase or decrease

We know,

$$E = \frac{\rho_L}{2\pi\epsilon_0\rho}\ a_\rho$$

We also know, work done

$$W = -Q\int_{initial}^{final} E.dL$$

$$= -Q\int_{\rho_1}^{\rho_2}\left(\frac{\rho_L}{2\pi\epsilon_0\rho}\ a_\rho\right)\cdot\left(d\rho\ a_\rho + \rho d\varphi\ a_\varphi + dz\ a_z\right)$$

$$= -\frac{Q\rho_L}{2\pi\epsilon_0}\int_{\rho_1}^{\rho_2}\frac{d\rho}{\rho}$$

$$= -\frac{Q\rho_L}{2\pi\epsilon_0}\ [\ln\rho]_{\rho_1}^{\rho_2}$$

$$W = -\frac{Q\rho_L}{2\pi\epsilon_0}\ \ln\frac{\rho_2}{\rho_1}$$

Work done when displacement $(\boldsymbol{\rho})$ is same but, change in angle $(\boldsymbol{\varphi})$.

We know,

$$E = \frac{\rho_L}{2\pi\epsilon_0\rho}\ a_\rho$$

We also know, work done

$$W = -Q\int_{initial}^{final} E.dL$$

$$= -Q\int_{\rho_1}^{\rho_2}\left(\frac{\rho_L}{2\pi\epsilon_0\rho}\ a_\rho\right)\cdot dL$$

Here, $dL = d\rho\ a_\rho + \rho d\varphi\ a_\varphi + dz\ a_z$. Change in agnle (φ) makes (ρ) and (z) $zero$.

$$= -Q\int_{\rho_1}^{\rho_2}\left(\frac{\rho_L}{2\pi\epsilon_0\rho}\ a_\rho\right)\cdot\left(\rho d\varphi\ a_\varphi\right)$$

$$W = 0$$

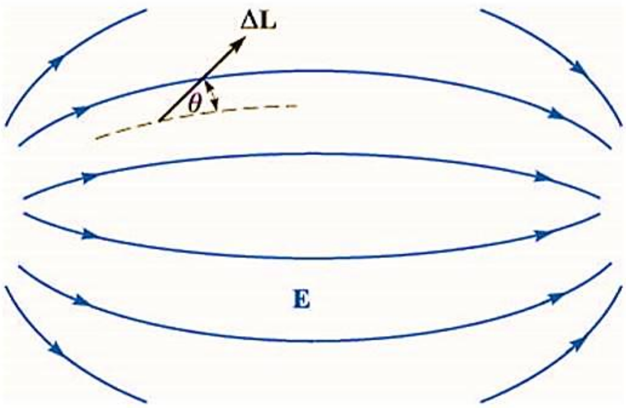
Potential Gradient (W.H.Hayt)

We know

$$V = -\int E \cdot dL$$

For a very short element of length ΔL along which E is constant, leading to an incremental potential difference ΔV,

$$\Delta V = -E \cdot \Delta L$$



If we designate the angle between ΔL and E as θ, then

$$\Delta V = -E \cdot \Delta L \cos\theta$$

Then,

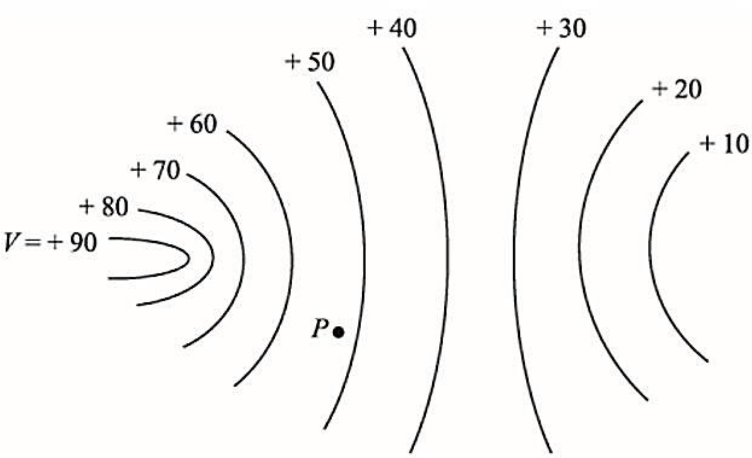
$$\frac{dV}{dL} = -E \cos\theta$$

It is obvious that the maximum positive increment of potential, ΔV_{max}, will occur when $\cos\theta$ is -1, or ΔL points in the direction opposite to E. For this condition,

$$\left.\frac{dV}{dL}\right|_{max} = E$$

Characteristics of the relationship between E and V:

- The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.
- This maximum value is obtain when the direction of the distance increment is opposite to E.



*Equipotential surfaces shown as lines in the two dimensional sketch.

At P, small incremental distance ΔL in various directions, to find that direction in which the potential is changing the most rapidly. From the figure this direction appears to be left and slightly upward. So the electric field intensity is therefore oppositely directed (to the right and slightly downward at P). Its magnitude is given by dividing the small increase in potential by the small element of length.

The direction in which the potential is increasing the most rapidly is perpendicular to the equipotentials (in the direction of increasing potential). If ΔL is directed along an equipotential, $\Delta V = 0$. Then,

$$\Delta V = -E \cdot \Delta L = 0$$

Since neither $E\ nor\ \Delta L\ is\ zero$, E must be perpendicular to ΔL or equipotentials.

Now, by letting a_N be a unit vector normal to the equipotential surface and directed toward the higher potentials. The electric field intensity is then expressed in terms of the potential,

$$E = -\left.\frac{dV}{dL}\right|_{max} a_N$$

The magnitude of E is given by the maximum space rate of change of V and the direction of E is normal to the equipotential surface (in the direction of decreasing potential).

Since $dV/dL|_{max}$ occurs when ΔL is in the direction of a_N,

$$\left.\frac{dV}{dL}\right|_{max} = \frac{dV}{dN}$$

$$E = -\frac{dV}{dN} a_N$$

The operation on V by which –E is obtained is known as the gradient and the gradient of a scalar field T is defined as

$$Gradient\ of\ T = grad\ T = \frac{dT}{dN}\ a_N$$

Using the new term, we now may write

$$E = -grad\ V$$

Now,

$$dV = \frac{\delta V}{\delta x}dx + \frac{\delta V}{\delta y}dy + \frac{\delta V}{\delta z}dz$$

Also,

$$V = -\int E \cdot dL$$

$$dV = -E \cdot dL = -E_x dx - E_y dy - E_z dz$$

Since both expressions are true for any $dx, dy\ and\ dz$, then

$$E_x = -\frac{\delta V}{\delta x}$$

$$E_y = -\frac{\delta V}{\delta y}$$

$$E_z = -\frac{\delta V}{\delta z}$$

Then,

$$E = -\left(\frac{\delta V}{\delta x}a_x + \frac{\delta V}{\delta y}a_y + \frac{\delta V}{\delta z}a_z\right)$$

$$E = -\left(\frac{\delta}{\delta x}a_x + \frac{\delta}{\delta y}a_y + \frac{\delta}{\delta z}a_z\right) \cdot V$$

$$\therefore E = -\nabla \cdot V \qquad where, \left[\nabla = \frac{\delta}{\delta x}a_x + \frac{\delta}{\delta y}a_y + \frac{\delta}{\delta z}a_z\right]$$

$$\nabla V = \frac{\delta V}{\delta x}a_x + \frac{\delta V}{\delta y}a_y + \frac{\delta V}{\delta z}a_z \quad (Cartesian)$$

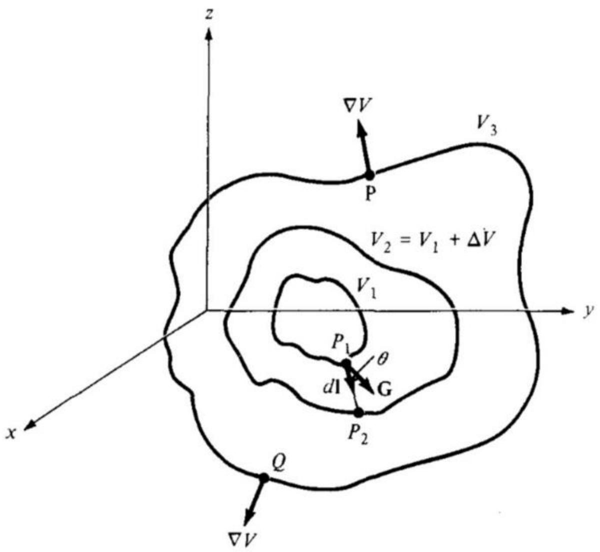
$$\nabla V = \frac{\delta V}{\delta \rho}a_\rho + \frac{1}{\rho}\ \frac{\delta V}{\delta \varphi}a_\varphi + \frac{\delta V}{\delta z}a_z \quad (Cylindrical)$$

$$\nabla V = \frac{\delta V}{\delta r}a_r + \frac{1}{r}\ \frac{\delta V}{\delta \theta}a_\theta + \frac{1}{r\ sin\theta}\ \frac{\delta V}{\delta \varphi}a_\varphi \quad (Spherical)$$

Gradient of a Scalar (Sadiku)

The gradient of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

Mathematical expression for the gradient can be obtained by evaluating the difference in the field dV between points $P_1 and\ P_2$ of following figure,



$$dv = \frac{\delta V}{\delta x}dx + \frac{\delta V}{\delta y}dy + \frac{\delta V}{\delta z}dz$$

$$= \left(\frac{\delta V}{\delta x}\ a_x + \frac{\delta V}{\delta y}\ a_y + \frac{\delta V}{\delta z}\ a_z\right) \cdot \left(dx\ a_x + dy\ a_y + dz\ a_z\right)$$

For convenience, let

$$G = \frac{\delta V}{\delta x}\ a_x + \frac{\delta V}{\delta y}\ a_y + \frac{\delta V}{\delta z}\ a_z$$

Then

$$dV = G \cdot dl$$

$$dV = G\ cos\theta\ dl$$

Or

$$\frac{dV}{dl} = G\ cos\theta$$

Where dl is the differential displacement from $P_1\ to\ P_2\ and\ \theta$ is the angle between $G\ and\ dl.$

dV/dl is maximum when $\theta = 0$, that is when dl is in the direction of G. Hence,

$$\left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dn} = G$$

Where, dV/dn is the normal derivative. Thus G has its magnitude and direction as those of the maximum rate of change of V. By definition, G is the gradient of V. Therefore,

$$grad\ V = \nabla V = \frac{\delta V}{\delta x}\ a_x + \frac{\delta V}{\delta y}\ a_y + \frac{\delta V}{\delta z}\ a_z$$

The gradient of V can be expressed in Cartesian, cylindrical and spherical coordinates.

$$\nabla V = \frac{\delta V}{\delta x} a_x + \frac{\delta V}{\delta y} a_y + \frac{\delta V}{\delta z} a_z \quad (Cartesian)$$

$$\nabla V = \frac{\delta V}{\delta \rho} a_\rho + \frac{1}{\rho}\ \frac{\delta V}{\delta \varphi} a_\varphi + \frac{\delta V}{\delta z} a_z \quad (Cylindrical)$$

$$\nabla V = \frac{\delta V}{\delta r} a_r + \frac{1}{r}\ \frac{\delta V}{\delta \theta} a_\theta + \frac{1}{r\ sin\theta}\ \frac{\delta V}{\delta \varphi} a_\varphi \quad (Spherical)$$

Mathematical problem-2:

Given,

$$v = 100\, r^2\, sin\theta$$

$$Electric\ field, E = ?$$

Solution:

We know,

$$E = -\nabla \cdot V$$

$$\Rightarrow\ -\left(\frac{\delta}{\delta r} a_r + \frac{1}{r}\ \frac{\delta}{\delta \theta} a_\theta + \frac{1}{r\ sin\theta}\ \frac{\delta}{\delta \varphi} a_\varphi\right) \cdot (100\, r^2\, sin\theta)$$

$$\Rightarrow\ -\frac{\delta(100\, r^2\, sin\theta)}{\delta r} a_r - \frac{\delta(100\, r^2\, sin\theta)}{r\ \delta r} a_\theta - 0$$

$$\Rightarrow\ -(100\, sin\theta)\ \frac{\delta}{\delta r}\ (r)^2\, a_r - 100r\ \frac{\delta}{\delta \theta}\ (sin\theta)\, a_\theta$$

$$\therefore E = -100\, sin\theta\, (2r)\, a_r - 100r\, cos\theta\, a_\theta \quad (Ans)$$

Mathematical problem-3:

Given,

$$v = 100\, \rho^2$$

$$Electric\ field, E = ?$$

Solution:

We know,

$$E = -\nabla \cdot V$$

$$\Rightarrow\ -\left(\frac{\delta}{\delta \rho} a_\rho + \frac{1}{\rho}\ \frac{\delta}{\delta \varphi} a_\varphi + \frac{\delta}{\delta z} a_z\right) \cdot (100\, \rho^2)$$

$$\Rightarrow\ -\frac{\delta\, (100\, \rho^2)}{\delta \rho} a_\rho - 0 - 0$$

$$\Rightarrow -100\, (2\rho)\, a_\rho$$

$$\therefore E = -200\rho\, a_\rho \quad (Ans)$$

Mathematical problem-4:

Potential field, $V = 2x^2y - 5z$ and a point $P(-4, 3, 6)$. Find following numerical values at point P:

(a) The potential, V
(b) The electric field intensity, E
(c) The direction of E
(d) The electric flux density D and
(e) The volume charge density ρ_v

Solution:

(a) The potential at $P(-4, 3, 6)$

$$V_P = 2(-4)^2(3) - 5(6) = 66\ V$$

(b) The electric field intensity

$$E = -\nabla V$$

$$E = -4xy\ a_x - 2x^2\ a_y + 5\ a_z \quad V/m$$

The value of E at point P is

$$E_P = 48\ a_x - 32\ a_y + 5\ a_z \quad V/m$$

And

$$|E_P| = \sqrt{48^2 + (-32)^2 + 5^2}$$

$$|E_P| = 57.9\ \ V/m$$

(c) The direction of E at point P is given by the unit vector

$$a_{E,P} = \frac{48\ a_x - 32\ a_y + 5\ a_z}{57.9}$$

$$a_{E,P} = 0.829\ a_x - 0.553\ a_y + 0.086\ a_z$$

(d) Assuming these fields exist in free space, then the electric flux density

$$D = \epsilon_0 E = -35.4\ xy\ a_x - 17.71\ x^2\ a_y + 44.3\ a_z\ \ pC/m^3$$

(e) The volume charge density

$$\rho_v = \nabla \cdot D$$

$$\rho_v = -35.4\ y\ \ pC/m^3$$

At P,

$$\rho_v = -106.2\ \ pC/m^3$$

The Dipole

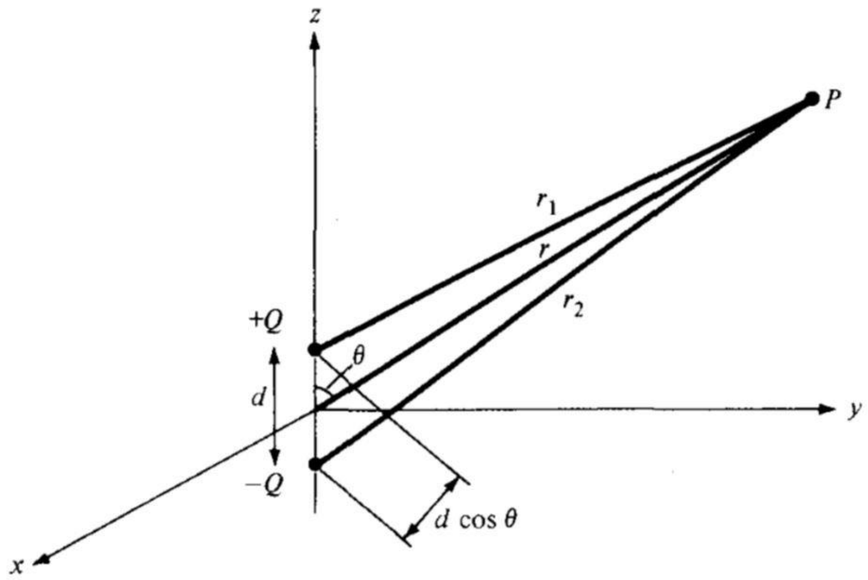
An electric dipole is formed when two point charges of equal magnitude but opposite sign are separated by a small distance.

Consider the dipole shown in following figure, the potential at point $P\ (r, \theta, \varphi)$ is

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r^1} - \frac{1}{r^2}\right]$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

Where, r_1 and r_2 are the distances between P and $+Q$ and P and $-Q$ respectively.



$$If\ r \gg d, r_2 - r_1 \simeq d\ cos\theta\ and\ r_2 r_1 \simeq r^2\ then\ we\ have,$$

$$V = \frac{Q}{4\pi\epsilon_0}\frac{d\ cos\theta}{r^2}$$

Electric field intensity,

$$E = -\nabla \cdot V$$

$$E = -\left(\frac{\delta V}{\delta r}a_r + \frac{1}{r}\frac{\delta V}{\delta\theta}a_\theta + \frac{1}{r\ sin\theta}\frac{\delta V}{\delta\varphi}a_\varphi\right)$$

Then

$$E = -\left(-\frac{Qd\ cos\theta}{4\pi\epsilon_0 r^3}a_r - \frac{Qd\ sin\theta}{4\pi\epsilon_0 r^3}a_\theta\right)$$

$$E = -\frac{Qd}{4\pi\epsilon_0}\left(-2\ cos\theta\frac{1}{r^3}\ a_r - sin\theta\frac{1}{r^3}\ a_\theta\right)$$

$$E = -\frac{Qd}{4\pi\epsilon_0 r^3}(-2\ cos\theta\ a_r - sin\theta\ a_\theta)$$

The potential field of the dipole may be simplified by making use of the dipole moment. The vector length directed from $-Q$ to $+Q$ as d and then define the dipole moment as Qd and assighn it the symbol P. Thus

$$p = Qd$$

The units of P are Cm.

Since $d.a_r = d\,cos\theta$, then we have

$$V = \frac{Q}{4\pi\epsilon_0}\frac{d\,cos\theta}{r^2}$$

$$V = \frac{p.a_r}{4\pi\epsilon_0 r^2}$$

This may be generalized as

$$V = \frac{1}{4\pi\epsilon_0\,|r - r'|^2}\;p\;\cdot\;\frac{r - r'}{|r - r'|}$$

$$V = \frac{p \cdot (r - r')}{4\pi\epsilon_0\,|r - r'|^3}$$

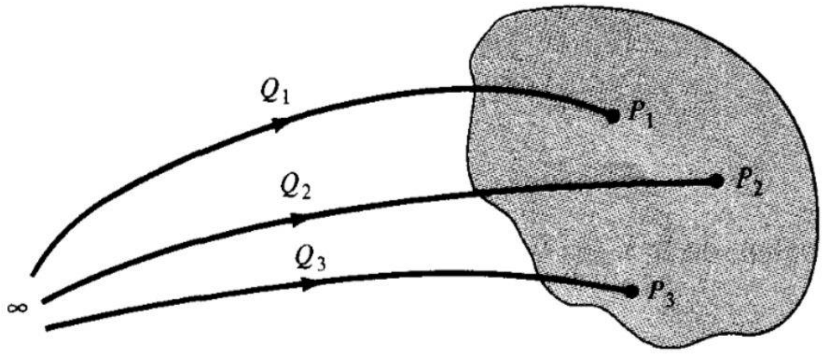
Energy Density in Electrostatic Fields

Three point charges $Q_1, Q_2\ and\ Q_3$ in an empty space shown in following figure.

- No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field.
- The work done in transferring of Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1.
- Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32} + V_{31})$, where V_{32} and V_{31} are the potentials at P_3 due to Q_2 and Q_1 respectively.

Work to position $Q_2 = \ Q_2\ V_{21}$

Work to position $Q_3 = \ Q_3\ (V_{31} + V_{32})$



The total work done in positioning the three charges is,

$$W_E = W_1 + W_2 + W_3$$

$$W_E = 0 + Q_2V_{21} + Q_3\ (V_{31} + V_{32}) \_\_\_\_\_\_\_\_\_\_\_\_ (1)$$

If the charges were positioned in reverse order, then,

$$W_E = W_3 + W_2 + W_1$$

$$W_E = 0 + Q_2V_{23} + Q_1\ (V_{12} + V_{13}) \_\_\_\_\_\_\_\_\_\_\_\_ (2)$$

Where, V_{23} is the potential at P_2 due to Q_3, V_{12} and V_{13} are respectively the potentials at P_1 due to Q_2 and Q_3.

Now adding equations (1) and (2) gives,

$$2W_E = Q_1\ (V_{12} + V_{13}) + Q_2\ (V_{21} + V_{23}) + Q_3\ (V_{31} + V_{32})$$

$$2W_E = Q_1\ V_1 + Q_2\ V_2 + Q_3\ V_3 \qquad [Here, V_{12} + V_{13} = V_1]$$

$$W_E = \frac{1}{2} Q_1\ V_1 + Q_2\ V_2 + Q_3\ V_3$$

Where, V_1, V_2 *and* V_3 are total potentials at P_1, P_2 *and* P_3 respectively. In general, if there are n point charges then,

$$W_E = \frac{1}{2}\sum_{m=1}^{m=n} Q_m V_m \qquad (in\ joules)$$

If, instead of point charges, the region has continuous charge distribution, then we have,

$$W_E = \frac{1}{2}\int \rho_L\, V\, dl \qquad (Line\ charge)$$

$$W_E = \frac{1}{2}\int \rho_S\, V\, dS \qquad (Surface\ charge)$$

$$W_E = \frac{1}{2}\int \rho_v\, V\, dv \qquad (Volume\ charge)$$

Since,

$$\rho_v = \nabla \cdot D$$

For volume charge, we can write

$$W_E = \frac{1}{2}\int_v (\nabla \cdot D)\, V\, dv$$

But for any vector A and scalar, the identity

$$\nabla \cdot VA = A \cdot \nabla V + V(\nabla \cdot A)$$

Or,

$$(\nabla \cdot A)V = \nabla \cdot VA - A \cdot \nabla V$$

Applying the identity, we get,

$$W_E = \frac{1}{2}\int_v (\nabla \cdot VD)\, dv - \frac{1}{2}\int_v (\mathrm{D} \cdot \nabla V)\, dv$$

Applying divergence theorem to the first term on the right-hand side of this equation, we have

$$W_E = \frac{1}{2}\oint_S (VD) \cdot dS - \frac{1}{2}\int_v (\mathrm{D} \cdot \nabla V)\, dv$$

As we know that

- V varies as $1/r$ and D as $1/r^2$ for point charges
- V varies as $1/r^2$ and D as $1/r^3$ for dipoles

So, here in the first term on the right-hand side of the equation

- VD must vary at least as $1/r^3$ and
- dS varies as r^2

Consequently, the first integral of the equation must tend to zero as the surface S becomes large. So the equations reduces to

$$W_E = -\frac{1}{2} \int_v (\mathrm{D} \cdot \nabla V)\, dv$$

Since, $E = -\nabla V$ and $D = \epsilon_0 E$, we have,

$$W_E = \frac{1}{2} \int \mathrm{D} \cdot \mathrm{E}\, dv$$

$$W_E = \frac{1}{2} \int (\epsilon_0 E \cdot \mathrm{E})\, dv$$

$$W_E = \frac{1}{2} \int \epsilon_0\, E^2\, dv$$

From this, we can define electrostatic energy density $W_E\ (in\, J/m^3)\ as$

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\, D \cdot E = \frac{1}{2}\, \epsilon_0\, E^2 = \frac{D^2}{2\epsilon_0}$$