CHAPTER-5

Conductors, Dielectrics and Capacitance

Current and Current Density

Electric charges in motion constitute a current. The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point of one coulomb per second. Current is symbolized by I and therefore

$$I = \frac{dQ}{dt}$$

Thus in a current of one ampere, charge is being transferred at a rate of one coulomb per second.

Now the concept of current density J, if current ΔI flows through a surface ΔS then the current density is

$$J_n = \frac{\Delta I}{\Delta S}$$

Or,

$$\Delta I = J_n \, \Delta S$$

Assuming that the current density is perpendicular to the surface. If the current density is not normal to the surface,

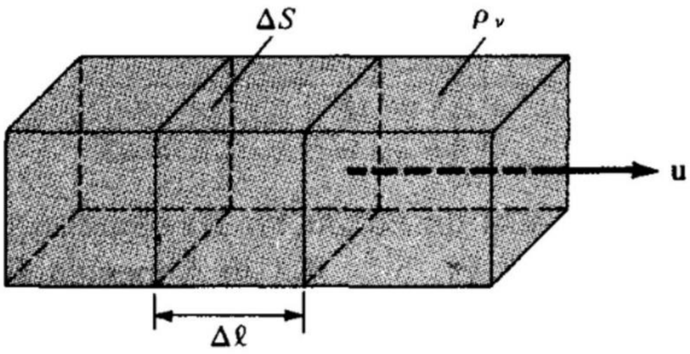
$$\Delta I = J \, \Delta S$$

Thus, the total current flowing through a surface S is

$$I = \int_S J \cdot dS$$

The current density at a given point is the current through a unit normal area at that point. Current density measured in amperes per square meter (A/m^2).

Charge in motion constitutes a current



Consider a filament of following figure. If there is a flow of charge of density ρ_v, at velocity $v = a_y\, \boldsymbol{a_y}$ then the current through the filament is

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

[$\boldsymbol{The\ element\ of\ charge\ \Delta Q = \rho_v\ \Delta S\ \Delta l}$]

$$\Delta I = \rho_v\ \Delta S\ \frac{\Delta l}{\Delta t}$$

$$\Delta I = \rho_v\ \Delta S\ v_y$$

The y-directed current density J_y is given by

$$J_y = \frac{\Delta I}{\Delta S}$$

$$J_y = \rho_v\ v_y$$

Hence, in general

$$J = \rho_v\ v$$

This result show very clearly that charge in motion constitutes a current.

Continuity of current

Due to the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume. Thus current I_{Out} coming out of the closed surface is

$$I_{Out} = \oint J \cdot dS$$

$$I_{Out} = \frac{-dQ_{in}}{dt} \quad \dots\dots\dots\dots\dots (1)$$

Where, Q_{in} is the total charge enclosed by the closed surface. Invoking divergence theorem

$$\oint_S J \cdot dS = \int_v \nabla \cdot J \, dv \quad \dots\dots\dots\dots\dots (2)$$

But

$$\frac{-dQ_{in}}{dt} = -\frac{d}{dt}\int_v \rho_v \, dv$$

$$\frac{-dQ_{in}}{dt} = -\int_v \frac{\delta \rho_v}{\delta t} dv \dots\dots\dots\dots\dots (3)$$

Now substituting equations (2) and (3) into equation (1) gives,

$$\int_v \nabla \cdot J \, dv = -\int_v \frac{\delta \rho_v}{\delta t} dv$$

Or

$$\nabla \cdot J = -\frac{\delta \rho_v}{\delta t}$$

Which is called the continuity of current equation.

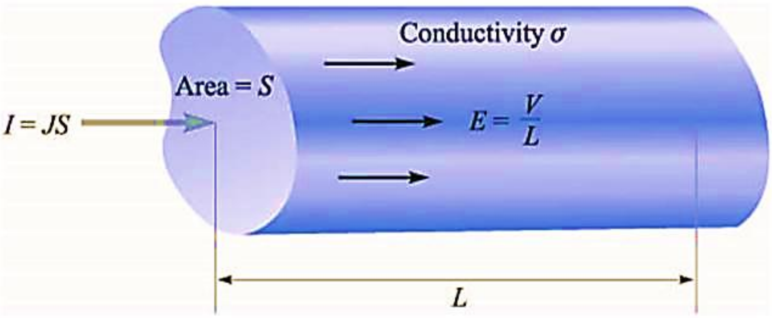
Resistance of Conductors of uniform fields

The relationship between J and E for a metallic conductor is specified by the conductivity σ ($Sigma$),

$$J = \sigma E$$

Where σ is measured in Siemens per meter (S/m). One Siemens (1 S) is the basic unit of conductance in the SI system and is defined as one ampere per volt.

Let us assume that J and E are uniform, as they are in the cylindrical region shown in following figure.



Since they are uniform,

$$I = \int_S J \cdot dS$$

$$I = J\, S$$

And

$$V_{ab} = -\int_b^a E \cdot dL$$

$$V_{ab} = -E \int_b^a dL$$

$$V_{ab} = -E \cdot L_{ba}$$

$$V_{ab} = -E \cdot L_{ab}$$

Or

$$V = E\, L$$

Thus

$$J = \frac{I}{S}$$

$$J = \sigma\, E$$

$$J = \sigma\, \frac{V}{L}$$

Or

$$V = \frac{L}{\sigma S}\, I$$

The ratio of the potential difference between the two ends of the cylinder to the current entering the more positive end, however, is recognized as the resistance of the cylinder and therefore,

$$V = I\,R$$

Where,

$$R = \frac{L}{\sigma S}$$

It is the resistance of any conductor of uniform cross section. If the cross section of the conduction is not uniform then this equation is not applicable.

However, the basic definition of resistance R as the ration of the potential difference V between the two ends of the conductor to the current I through the conductor still applies. The general expression for resistance of a conductor of non-uniform cross section,

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a E \cdot dL}{\int_S \sigma E \cdot dS}$$

Mathematical Problem-1:

Evaluated the resistance of a 1-mile length of copper wire, which has a diameter of 0.0508 inch.

Solution:

$$Diameter\ of\ the\ wire = 0.0508\ inch$$

$$= 0.0508 \times 0.0254$$

$$= 1.291 \times 10^{-3}\ meter$$

$$The\ area\ of\ the\ cross\ section = \pi r^2$$

$$= \pi \times \left(\frac{1.291 \times 10^{-3}}{2}\right)^2$$

$$= 1.308 \times 10^{-6}\ m^2$$

$$The\ length = 1\ mile = 1609\ meter$$

$$Conductivity, \sigma = 5.80 \times 10^7\ S/m$$

The resistance of the wire is therefore,

$$R = \frac{L}{\sigma S}$$

$$R = \frac{1609}{(5.80 \times 10^7) \times (1.308 \times 10^{-6})}$$

$$R = 21.2\ \Omega \quad \text{(Ans)}$$

Boundary conditions

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called boundary conditions. These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

We shall consider the boundary conditions at an interface separating

- Dielectric (ϵ_{r1}) and dielectric (ϵ_{r2})
- Conductor and dielectric
- Conductor and free space

To determine the boundary conditions, we need to use Maxwell's equations:

$$\oint E \cdot dl = 0$$

And

$$\oint D \cdot dS = Q_{enclosed}$$

Also we need to decompose the electric field intensity E into two orthogonal components,

$$E = E_t + E_n$$

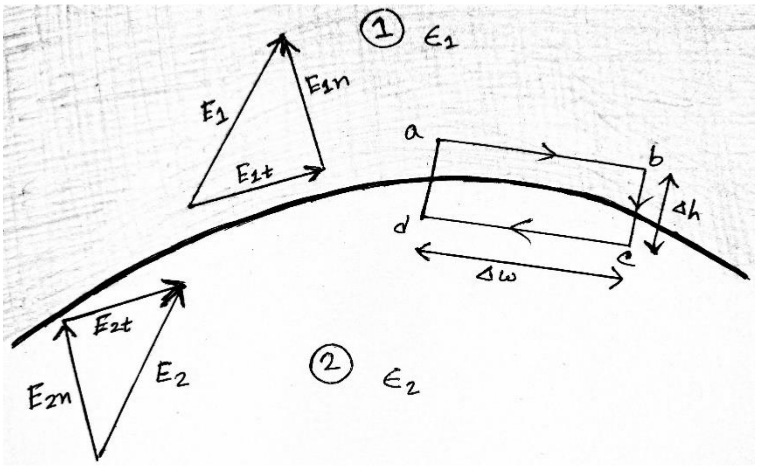
Where, E_t *and* E_n are, respectively, the tangential and normal components of E to the interface of interest. Similar decomposition can be done for the electric flux density D.

Dielectric-Dielectric Boundary Conditions

Consider the E field existing in a region consisting of two different dielectrics characterized by $\epsilon_1 = \epsilon_0\epsilon_{r1}\ and\ \epsilon_2 = \epsilon_0\epsilon_{r2}$ as shown in following figure. $E_1\ and\ E_2$ in media 1 and 2 respectively, can be decomposed as

$$E_1 = E_{1t} + E_{1n} \dots\dots\dots\dots (1)$$

$$E_2 = E_{2t} + E_{2n} \dots\dots\dots\dots (2)$$



Now, applying $\oint E \cdot dl = 0$ to the closed path $abcda$ assuming that the path is very small with respect to the variation of E. We get

$$0 = E_{1t}\,\Delta w - E_{1n}\,\frac{\Delta h}{2} - E_{2n}\,\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\,\frac{\Delta h}{2} + E_{1n}\,\frac{\Delta h}{2} \dots\dots\dots\dots (3)$$

Where

$$E_t = |E_t|\ and\ E_n = |E_n|.$$

As $\Delta h \to 0$, then equation (3) becomes

$$E_{1t} = E_{2t} \dots\dots\dots\dots (4)$$

Thus the tangential components of E are the same on the two sides of the boundary. In other words, E_t undergoes no change on the boundary and it is said to be continuous across the boundary.

Since $D = \epsilon E = D_t + D_n$, then equation (4) can be written as

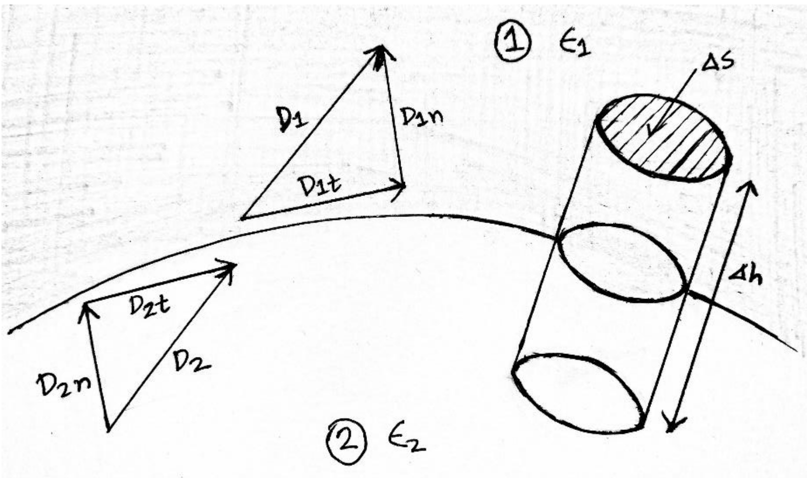
$$\frac{D_{1t}}{\epsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\epsilon_2}$$

Or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2} \dots\dots\dots\dots (5)$$

D_t undergoes some change across the interface. Hence D_t is said to be discontinuous across the interface.

Similarly, we apply $\oint D \cdot dS = Q_{enclosed}$ to the pillbox (Gaussian surface) of following figure.



Allowing $\Delta h \to 0$ gives

$$\Delta Q = \rho_S \, \Delta S$$

$$\Delta Q = D_{1n} \, \Delta S - D_{2n} \, \Delta S$$

Or

$$D_{1n} - D_{2n} = \rho_S \dots \dots \dots \dots (6)$$

Where ρ_S is the free charge density placed deliberately at the boundary. Equation (6) is based on the assumption that D is directed from region 2 to region 1. If no free charge exist at the interface then $\rho_S = 0$ and equation (6) becomes

$$D_{1n} = D_{2n} \dots \dots \dots \dots (7)$$

Thus the normal component of D is continuous across the interface; that is, D_n undergoes no change at the boundary.

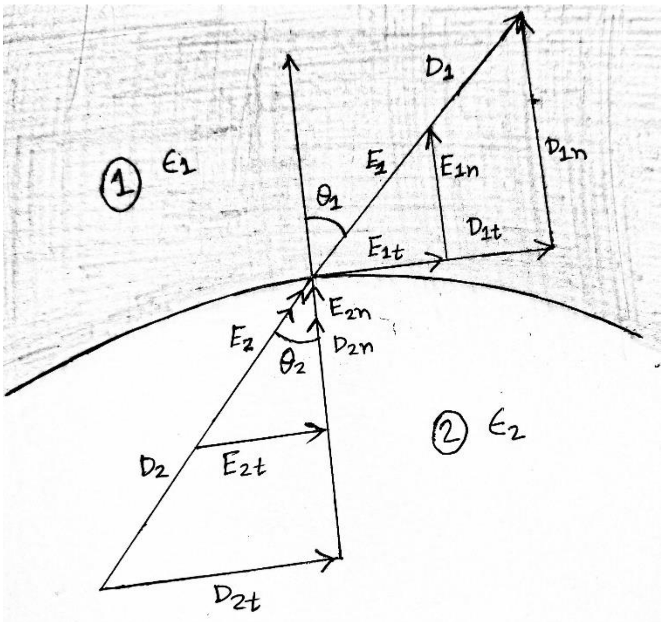
Since $D = \epsilon E$ then equation (7) can be written as

$$\epsilon_1 \, E_{1n} = \epsilon_2 \, E_{2n} \dots \dots \dots \dots (8)$$

The normal component of E is discontinuous at the boundary.

Equation (4) and (7) are collectively referred to as boundary conditions; they must be satisfied by an electric field at the boundary separating two different dielectrics.

We can also use the boundary conditions to determine the refraction of the electric field across the interface. Consider D_1 or E_1 and D_2 or E_2 making angles θ_1 and θ_2 with normal to the interface as illustrated in following figure.



Using equation (4), we have

$$E_1\ sin\theta_1 = E_{1t} = E_{2t} = E_2\ sin\theta_2$$

$$E_1\ sin\theta_1 = E_2\ sin\theta_2 \ldots\ldots\ldots\ldots (9)$$

Similarly, by applying equation (7) or (8), we get

$$\epsilon_1\ E_1\ cos\theta_1 = D_{1n} = D_{2n} = \epsilon_2 E_2\ cos\theta_2$$

Or

$$\epsilon_1\ E_1\ cos\theta_1 = \epsilon_2 E_2\ cos\theta_2 \ldots\ldots\ldots\ldots (10)$$

Now, dividing equation (9) by equation (10) gives,

$$\frac{\tan\theta_1}{\epsilon_1} = \frac{tan\theta_2}{\epsilon_2} \ldots\ldots\ldots\ldots (11)$$

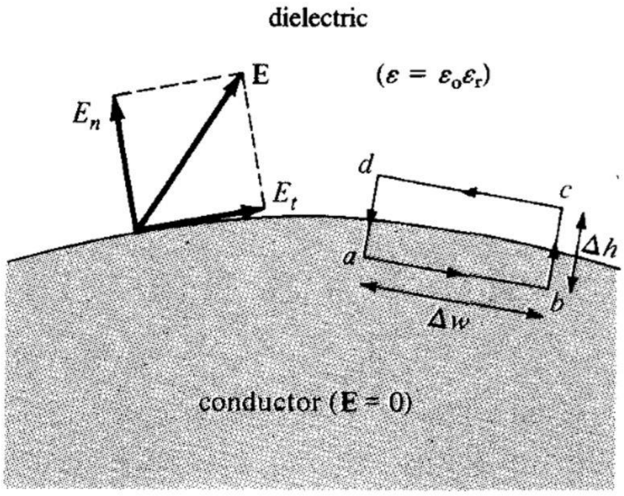
Since $\epsilon_1 = \epsilon_0\epsilon_{r1}$ and $\epsilon_2 = \epsilon_0\epsilon_{r2}$, then equation (11) becomes

$$\frac{\tan\theta_1}{tan\theta_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}$$

This is the law of refraction of the electric field at a boundary free of charge (Since $\rho_S = 0$ is assumed at the interface).

Conductor-Dielectric Boundary Conditions

This is the case shown in following figure. The conductor is assumed to be perfect $(i.e., \sigma \rightarrow \infty\ or\ \rho_C \rightarrow 0)$. Although such a conduction is not practically realizable, we may regard conductors such as copper and silver.



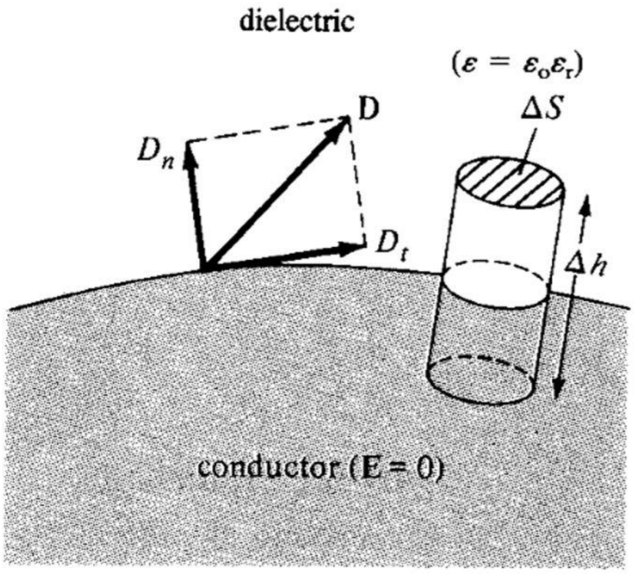
Applying $\oint E \cdot dl = 0$ to the closed path $abcda$ gives

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2} \ \ldots\ldots\ldots\ldots (1)$$

As $\Delta h \rightarrow 0$,

$$E_t = 0 \ \ldots\ldots\ldots\ldots (2)$$

Similarly, by applying $\oint D \cdot dS = Q_{enclosed}$ to the pillbox of following figure



Allowing $\Delta h \rightarrow 0$, we get

$$\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S \ \ldots\ldots\ldots\ldots (3)$$

As $D = \epsilon E = 0$ inside the conductor. Equation (3) may be written as

$$D_n = \frac{\Delta Q}{\Delta S} = \rho_S$$

Or

$$D_n = \rho_S \dots \dots \dots \dots (4)$$

Thus under static conditions,

- No electric field may exist within a conductor, that is

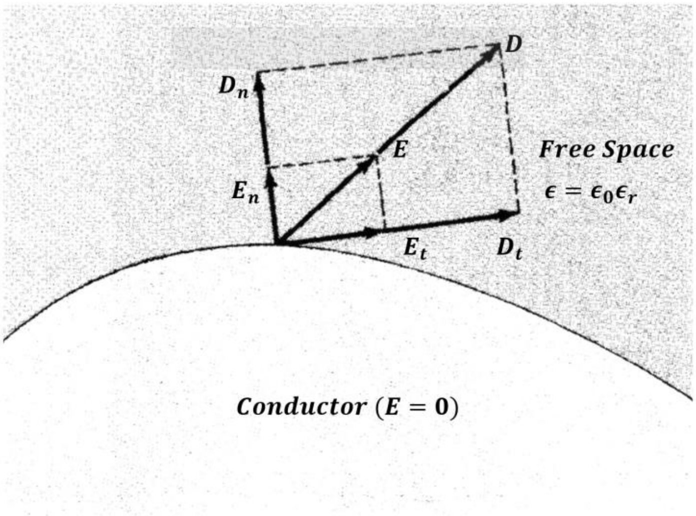
$$\rho_v = 0, \quad E = 0 \dots \dots \dots \dots (5)$$

- Since $E = -\nabla V = 0$, there can be no potential difference between any two points in the conductor.
- The electric field E can be external to the conductor and normal to its surface; that is

$$D_t = \epsilon_0 \epsilon_r E_t = 0, \quad D_n = \epsilon_0 \epsilon_r E_n = \rho_S \dots \dots \dots \dots (6)$$

Conductor-Free Space Boundary Conditions

This is a special case of the conductor-dielectric conditions and is illustrated in following figure.



The boundary conditions at the interface between a conductor and free space can be obtained from the following equation

$$D_t = \epsilon_0 \epsilon_r E_t = 0, \quad D_n = \epsilon_0 \epsilon_r E_n = \rho_S$$

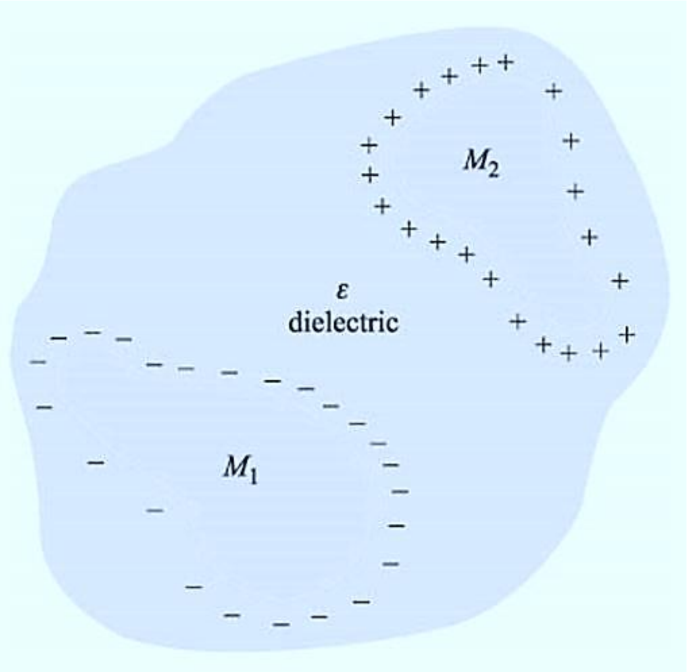
By replacing ϵ_r by 1 (because free space may be regarded as a special dielectric for which $\epsilon_r = 1$). We expect E to be external to the conductor and normal to its surface. Thus the boundary conditions are

$$D_t = \epsilon_0 E_t = 0, \quad D_n = \epsilon_0 E_n = \rho_S$$

E field must approach a conducting surface normally.

Capacitance

Two oppositely charged conductors M_1 and M_2 surrounded by a uniform dielectric. Let as designate the potential difference between M_2 and M_1 as V. Then we can define the capacitance as-



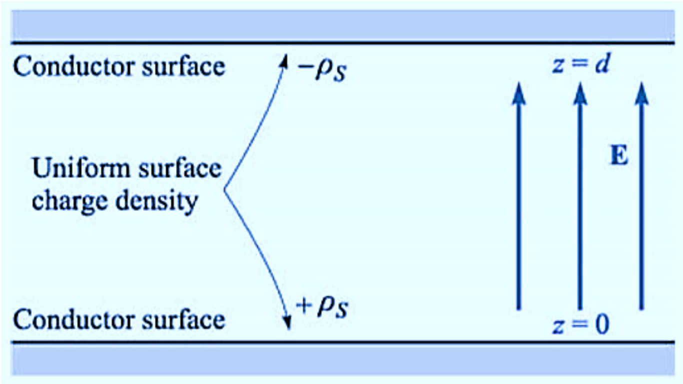
The ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between conductors; that is

$$C = \frac{Q}{V}$$

$$C = \frac{\oint \epsilon\, E \cdot dS}{-\int E \cdot dL}$$

The capacitance C is a physical property of the capacitor and in measured in Farads (F), where a farad is defined as one coulomb per volt.

Now applying the definition of capacitance to a simple two-conductor system shown in following figure in which the conductors are identical, infinite parallel planes with separation d.



The lower conducting plane at $z = 0$ and the upper one at $z = d$, a uniform sheet of surface charge $\pm\rho_S$ on each conductor leads to the uniform field

$$E = \frac{\rho_S}{\epsilon} a_z$$

Where the permittivity of the homogeneous dielectric is ϵ, and

$$D = \rho_S \, a_z$$

The charge on the lower plane must then be positive, since D is directed upward and the normal value of D is equal to the surface charge density there.

$$D_N = D_z = \rho_S$$

On the upper plane,

$$D_N = -D_z$$

And the surface charge there is the negative of that on the lower plane.

The potential difference between lower and upper planes is

$$V_0 = -\int_{upper}^{lower} E \cdot dL$$

$$V_0 = -\int_{d}^{0} \frac{\rho_S}{\epsilon} dz$$

$$V_0 = \frac{\rho_S}{\epsilon} d$$

Now,

$$Q = \rho_S \, S$$

$$V_0 = \frac{\rho_S}{\epsilon} d$$

Then

$$C = \frac{Q}{V_0}$$

$$C = \frac{\rho_S\ S}{\frac{\rho_S}{\epsilon}\ d}$$

$$C = \frac{\epsilon\ S}{d}$$

The total energy stored in the capacitor is

$$W_E = \frac{1}{2} \int_{vol} \epsilon\ E^2\ dv$$

$$W_E = \frac{1}{2} \int_0^S \int_0^d \left(\frac{\rho_S}{\epsilon}\right)^2 \epsilon\ dz\ dS$$

$$W_E = \frac{1}{2} \frac{{\rho_S}^2}{\epsilon^2} \frac{\epsilon \cdot d^2 \cdot S}{d}$$

$$W_E = \frac{1}{2} \frac{{\rho_S}^2}{\epsilon}\ Sd$$

$$W_E = \frac{1}{2} \frac{\epsilon\ S}{d} \frac{{\rho_S}^2 d^2}{\epsilon^2}$$

$$W_E = \frac{1}{2} \frac{\epsilon\ S}{d} \left(\frac{\rho_S}{\epsilon}\ d\right)^2$$

$$W_E = \frac{1}{2}\ CV^2$$

$$W_E = \frac{1}{2}\ C \left(\frac{Q}{C}\right)^2$$

$$W_E = \frac{1}{2} \frac{Q^2}{C}$$

Mathematical Problem-2:

Calculate the capacitance of a parallel-plate capacitor having a mica dielectric, $\epsilon_R = 6$, a plate area of 10 $inch^2$ and a separation of 0.01 inch.

Solution:

Given,

$$S = 10 \times 0.0254^2 = 6.45 \times 10^{-3}\ m^2$$

$$d = 0.01 \times 0.0254 = 2.54 \times 10^{-4}\ m$$

Therefore,

$$C = \frac{6 \times 8.854 \times 10^{-12} \times 6.45 \times 10^{-3}}{d2.54 \times 10^{-4}}$$

$$C = 1.349\ nF \quad (Ans)$$