Transformation

Solved Problems: Problem 01

Derive the transformation that rotates an object point θ° about the origin. Write the matrix representation for this rotation.

SOLUTION

Refer to Fig. 4-13. Definition of the trigonometric functions sin and cos yields

$$x' = r\cos(\theta + \phi) \qquad y' = r\sin(\theta + \phi)$$

and

$$x = r\cos\phi \qquad y = r\sin\phi$$

Using trigonometric identities, we obtain

$$r\cos(\theta + \phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi) = x\cos\theta - y\sin\theta$$

and

$$r\sin(\theta + \phi) = r(\sin\theta\cos\phi + \cos\theta\sin\phi) = x\sin\theta - y\cos\theta$$

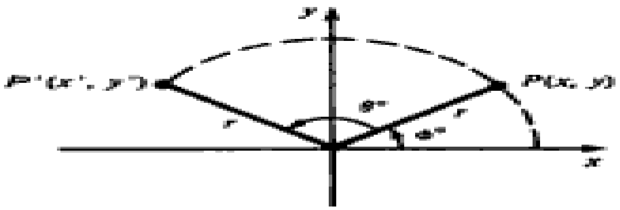
or

$$x' = x\cos\theta - y\sin\theta \qquad y' = x\sin\theta + y\cos\theta$$

Writing $P' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, $P = \begin{pmatrix} x \\ y \end{pmatrix}$, and

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

we can now write $P' = R_\theta \cdot P$.



Solved Problems: Problem 02

(*a*) Find the matrix that represents rotation of an object by 30° about the origin.
(*b*) What are the new coordinates of the point $P(2, -4)$ after the rotation?

SOLUTION

(*a*) From Prob. 4.1:

$$R_{30^\circ} = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(*b*) So the new coordinates can be found by multiplying:

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} \sqrt{3}+2 \\ 1-2\sqrt{3} \end{pmatrix}$$

Solved Problems: Problem 3

Write the general form of the matrix for rotation about a point $P(h, k)$.

SOLUTION

Following Prob. 4.3, we write $R_{\theta,P} = T_{\mathbf{v}} \cdot R_{\theta} \cdot T_{-\mathbf{v}}$, where $\mathbf{v} = h\mathbf{I} + k\mathbf{J}$. Using the 3 x 3 homogeneous coordinate form for the rotation and translation matrices, we have

$$R_{\theta,P} = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -h \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta) & -\sin(\theta) & [-h\cos(\theta) + k\sin(\theta) + h] \\ \sin(\theta) & \cos(\theta) & [-h\sin(\theta) - k\cos(\theta) + k] \\ 0 & 0 & 1 \end{pmatrix}$$

Solved Problems: Problem 4

Perform a 45° rotation of triangle $A(0,0)$, $B(1,1)$, $C(5,2)$ (*a*) about the origin and (*b*) about $P(-1,-1)$.

SOLUTION

We represent the triangle by a matrix formed from the homogeneous coordinates of the vertices:

$$\begin{matrix} A & B & C \end{matrix}$$
$$\begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

(*a*) The matrix of rotation is

$$R_{45^\circ} = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the coordinates $A'B'C'$ of the rotated triangle ABC can be found as

$$[A'B'C'] = R_{45^\circ} \cdot [ABC] = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \overset{\begin{matrix} A' & B' & C' \end{matrix}}{\begin{pmatrix} 0 & 0 & \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{7\sqrt{2}}{2} \\ 1 & 1 & 1 \end{pmatrix}}$$

Thus $A' = (0,0)$, $B' = (0, \sqrt{2})$, and $C' = (\frac{3}{2}\sqrt{2}, \frac{7}{2}\sqrt{2})$.

Solved Problems: Problem 4

(b) From Prob. 4.4, the rotation matrix is given by $R_{45^\circ,P} = T_{\mathbf{v}} \cdot R_{45^\circ} \cdot T_{-\mathbf{v}}$, where $\mathbf{v} = -\mathbf{I} - \mathbf{J}$. So

$$R_{45^\circ,P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{pmatrix}$$

Now

$$[A'B'C'] = R_{45^\circ,P} \cdot [ABC] = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & (\sqrt{2}-1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & (\frac{3}{2}\sqrt{2}-1) \\ (\sqrt{2}-1) & (2\sqrt{2}-1) & (\frac{9}{2}\sqrt{2}-1) \\ 1 & 1 & 1 \end{pmatrix}$$

So $A' = (-1, \sqrt{2}-1)$, $B' = (-1, 2\sqrt{2}-1)$, and $C' = (\frac{3}{2}\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1)$.

Solved Problems: Problem 5

Find the transformation that scales (with respect to the origin) by (*a*) *a* units in the *X* direction, (*b*) *b* units in the *Y* direction, and (*c*) simultaneously *a* units in the *X* direction and *b* units in the *Y* direction.

SOLUTION

(*a*) The scaling transformation applied to a point $P(x, y)$ produces the point (ax, y). We can write this in matrix form as $S_{a,1} \cdot P$, or

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ y \end{pmatrix}$$

(*b*) As in part (*a*), the required transformation can be written in matrix form as $S_{1,b} \cdot P$. So

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ by \end{pmatrix}$$

(*c*) Scaling in both directions is described by the transformation $x' = ax$ and $y' = by$. Writing this in matrix form as $S_{a,b} \cdot P$, we have

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

Solved Problems: Problem 6

Magnify the triangle with vertices $A(0, 0)$, $B(1, 1)$, and $C(5, 2)$ to twice its size while keeping $C(5, 2)$ fixed.

SOLUTION

From Prob. 4.7, we can write the required transformation with $\mathbf{v} = 5\mathbf{I} + 2\mathbf{J}$ as

$$S_{2,2,C} = T_{\mathbf{v}} \cdot S_{2,2} \cdot T_{-\mathbf{v}}$$
$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Representing a point P with coordinates (x, y) by the column vector $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, we have

$$S_{2,2,C} \cdot A = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$
$$S_{2,2,C} \cdot B = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$
$$S_{2,2,C} \cdot C = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

So $A' = (-5, -2)$, $B' = (-3, 0)$, and $C' = (5, 2)$. Note that, since the triangle ABC is completely determined by its vertices, we could have saved much writing by representing the vertices using a 3×3 matrix

$$[ABC] = \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

and applying $S_{2,2,C}$ to this. So

$$S_{2,2,C} \cdot [ABC] = \begin{pmatrix} 2 & 0 & -5 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -3 & 5 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} = [A'B'C']$$

Solved Problems: **Problem 7**

Find the form of the matrix for reflection about a line L with slope m and y intercept $(0, b)$.

SOLUTION

Following Prob. 4.9 and applying the fact that the angle of inclination of a line is related to its slope m by the equation $\tan(\theta) = m$, we have with $\mathbf{v} = b\mathbf{J}$,

$$M_L = T_{\mathbf{v}} \cdot R_\theta \cdot M_x \cdot R_{-\theta} \cdot T_{-\mathbf{v}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

Now if $\tan(\theta) = m$, standard trigonometry yields $\sin(\theta) = m/\sqrt{m^2+1}$ and $\cos(\theta) = 1/\sqrt{m^2+1}$. Substituting these values for $\sin(\theta)$ and $\cos(\theta)$ after matrix multiplication, we have

$$M_L = \begin{pmatrix} \dfrac{1-m^2}{m^2+1} & \dfrac{2m}{m^2+1} & \dfrac{-2bm}{m^2+1} \\ \dfrac{2m}{m^2+1} & \dfrac{m^2-1}{m^2+1} & \dfrac{2b}{m^2+1} \\ 0 & 0 & 1 \end{pmatrix}$$

Solved Problems:

Problem 8

Reflect the diamond-shaped polygon whose vertices are $A(-1,0)$, $B(0,-2)$, $C(1,0)$, and $D(0,2)$ about (*a*) the horizontal line $y=2$, (*b*) the vertical line $x=2$, and (*c*) the line $y=x+2$.

SOLUTION

We represent the vertices of the polygon by the homogeneous coordinate matrix

$$V=\begin{pmatrix}-1 & 0 & 1 & 0\\ 0 & -2 & 0 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}$$

From Prob. 4.9, the reflection matrix can be written as

$$M_L = T_{\mathbf{v}}\cdot R_\theta \cdot M_x \cdot R_{-\theta}\cdot T_{-\mathbf{v}}$$

(*a*) The line $y=2$ has y intercept $(0,2)$ and makes an angle of 0° with the x axis. So with $\theta=0$ and $\mathbf{v}=2\mathbf{J}$, the transformation matrix is

$$M_L=\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-2\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-1&4\\0&0&1\end{pmatrix}$$

This same matrix could have been obtained directly by using the results of Prob. 4.10 with slope $m=0$ and y intercept $b=2$. To reflect the polygon, we set

$$M_L\cdot V=\begin{pmatrix}1&0&0\\0&-1&4\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&1&0\\0&-2&0&2\\1&1&1&1\end{pmatrix}=\begin{pmatrix}\overset{A'}{-1}&\overset{B'}{0}&\overset{C'}{1}&\overset{D'}{0}\\4&6&4&2\\1&1&1&1\end{pmatrix}$$

Converting from homogeneous coordinates, $A'=(-1,4)$, $B'=(0,6)$, $C'=(1,4)$, and $D'=(0,2)$.

(*b*) The vertical line $x=2$ has no y intercept and an infinite slope! We can use M_y, reflection about the y axis, to write the desired reflection by (1) translating the given line two units over to the y axis, (2) reflect about the y axis, and (3) translate back two units. So with $\mathbf{v}=2\mathbf{I}$,

$$M_L = T_{\mathbf{v}}\cdot M_y\cdot T_{-\mathbf{v}}$$

$$=\begin{pmatrix}1&0&2\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&-2\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-1&0&4\\0&1&0\\0&0&1\end{pmatrix}$$

Finally

$$M_L\cdot V=\begin{pmatrix}-1&0&4\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&1&0\\0&-2&0&2\\1&1&1&1\end{pmatrix}=\begin{pmatrix}5&4&3&4\\0&-2&0&2\\1&1&1&1\end{pmatrix}$$

or $A'=(5,0)$, $B'=(4,-2)$, $C'=(3,0)$, and $D'=(4,2)$.

(*c*) The line $y=x+2$ has slope 1 and a y intercept $(0,2)$. From Prob. 4.10, with $m=1$ and $b=2$, we find

$$M_L=\begin{pmatrix}0&1&-2\\1&0&2\\0&0&1\end{pmatrix}$$

The required coordinates A', B', C', and D' can now be found.

$$M_L\cdot V=\begin{pmatrix}0&1&-2\\1&0&2\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&1&0\\0&-2&0&2\\1&1&1&1\end{pmatrix}=\begin{pmatrix}-2&-4&-2&0\\1&2&3&2\\1&1&1&1\end{pmatrix}$$

So $A'=(-2,1)$, $B'=(-4,2)$, $C'=(-2,3)$, and $D'=(0,2)$.

Solved Problems:

Problem 9

An observer standing at the origin sees a point $P(1, 1)$. If the point is translated one unit in the direction $\mathbf{v} = \mathbf{I}$, its new coordinate position is $P'(2, 1)$. Suppose instead that the observer stepped back one unit along the x axis. What would be the apparent coordinates of P with respect to the observer?

SOLUTION

The problem can be set up as a transformation of coordinate systems. If we translate the origin O in the direction $\mathbf{v} = -\mathbf{I}$ (to a new position at O') the coordinates of P in this system can be found by the translation $\bar{T}_{\mathbf{v}}$:

$$\bar{T}_{\mathbf{v}} \cdot P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

So the new coordinates are $(2, 1)'$. This has the following interpretation: a displacement of one unit in a given direction can be achieved by either moving the object forward or stepping back from it.

Solved Problems:

Problem 10

Find the equation of the circle $(x')^2 + (y')^2 = 1$ in terms of xy coordinates, assuming that the $x'y'$ coordinate system results from a scaling of a units in the x direction and b units in the y direction.

SOLUTION

From the equations for a coordinate scaling transformation, we find

$$x' = \frac{1}{a}x \qquad y' = \frac{1}{b}y$$

Substituting, we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Notice that as a result of scaling, the equation of the circle is transformed to the equation of an ellipse in the xy coordinate system.

Solved Problems:

Problem 11

Find the equation of the line $y' = mx' + b$ in xy coordinates if the $x'y'$ coordinate system results from a $90°$ rotation of the xy coordinate system.

SOLUTION

The rotation coordinate transformation equations can be written as

$$x' = x\cos(90^\circ) + y\sin(90^\circ) = y \qquad y' = -x\sin(90^\circ) + y\cos(90^\circ) = -x$$

Substituting, we find $-x = my + b$. Solving for y, we have $y = (-1/m)x - b/m$.

Solved Problems: Problem 12

Define *tilting* as a rotation about the x axis followed by a rotation about the y axis: (a) find the tilting matrix; (b) does the order of performing the rotation matter?

SOLUTION

(a) We can find the required transformation T by composing (concatenating) two rotation matrices:

$$T = R_{\theta_y,\mathbf{J}} \cdot R_{\theta_x,\mathbf{I}}$$

$$= \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_y & \sin\theta_y \sin\theta_x & \sin\theta_y \cos\theta_x & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ -\sin\theta_y & \cos\theta_y \sin\theta_x & \cos\theta_y \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) We multiply $R_{\theta_x,\mathbf{I}} \cdot R_{\theta_y,\mathbf{J}}$ to obtain the matrix

$$\begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ \sin\theta_x \sin\theta_y & \cos\theta_x & -\sin\theta_x \cos\theta_y & 0 \\ -\cos\theta_x \sin\theta_y & \sin\theta_x & \cos\theta_x \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is not the same matrix as in part a; thus the order of rotation matters.

Solved Problems:

Problem 13

Find a transformation A_V which aligns a given vector **V** with the vector **K** along the positive z axis.

SOLUTION

See Fig. 6-4(a). Let $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. We perform the alignment through the following sequence of transformations [Figs. 6-4(b) and 6-4(c)]:

1. Rotate about the x axis by an angle θ_1 so that V rotates into the upper half of the xz plane (as the vector $\mathbf{V}_1$).
2. Rotate the vector $\mathbf{V}_1$ about the y axis by an angle $-\theta_2$ so that $\mathbf{V}_1$ rotates to the positive z axis (as the vector $\mathbf{V}_2$).

Implementing step 1 from Fig. 6-4(b), we observe that the required angle of rotation θ_1 can be found by looking at the projection of **V** onto the yz plane. (We assume that b and c are not both zero.) From triangle $OP'B$:

$$\sin\theta_1 = \frac{b}{\sqrt{b^2+c^2}} \qquad \cos\theta_1 = \frac{c}{\sqrt{b^2+c^2}}$$

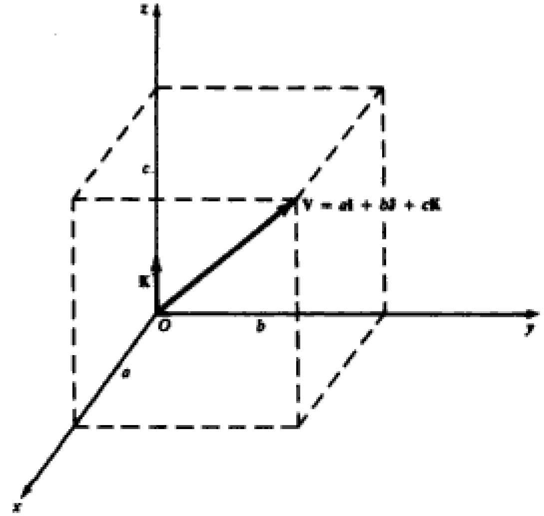
The required rotation is

$$R_{\theta_1,\mathbf{I}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{\sqrt{b^2+c^2}} & -\frac{b}{\sqrt{b^2+c^2}} & 0 \\ 0 & \frac{b}{\sqrt{b^2+c^2}} & \frac{c}{\sqrt{b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

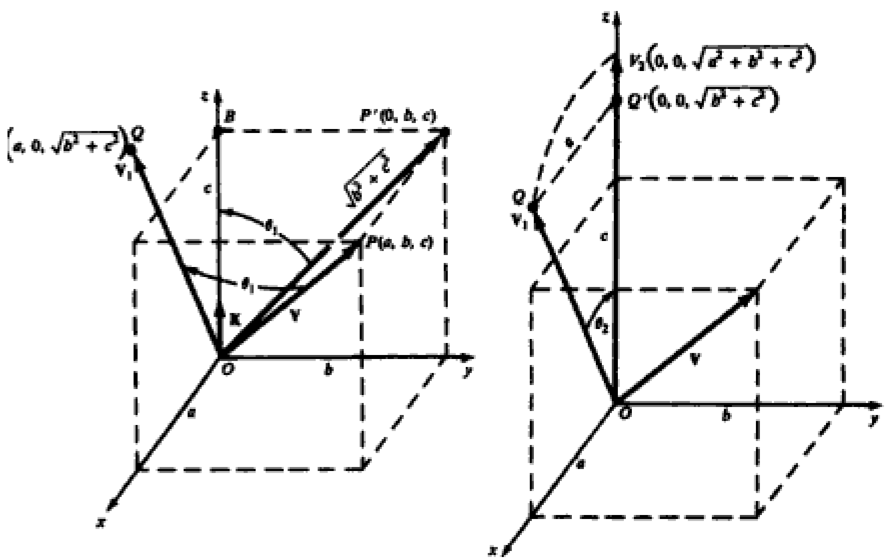
Applying this rotation to the vector **V** produces the vector $\mathbf{V}_1$ with the components $(a, 0, \sqrt{b^2+c^2})$.

Solved Problems:

Problem 13..



(a)



(b)

(c)

Solved Problems:

Problem 13..

Implementing step 2 from Fig. 6-4(*c*), we see that a rotation of $-\theta_2$ degrees is required, and so from triangle OQQ':

$$\sin(-\theta_2) = -\sin\theta_2 = -\frac{a}{\sqrt{a^2+b^2+c^2}} \qquad \text{and} \qquad \cos(-\theta_2) = \cos\theta_2 = \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}$$

Then

$$R_{-\theta_2,J} = \begin{pmatrix} \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 & \frac{-a}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{\sqrt{a^2+b^2+c^2}} & 0 & \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $|\mathbf{V}| = \sqrt{a^2+b^2+c^2}$, and introducing the notation $\lambda = \sqrt{b^2+c^2}$, we find

$$A_{\mathbf{V}} = R_{-\theta_2,J} \cdot R_{\theta_1,I}$$

$$= \begin{pmatrix} \frac{\lambda}{|\mathbf{V}|} & \frac{-ab}{\lambda|\mathbf{V}|} & \frac{-ac}{\lambda|\mathbf{V}|} & 0 \\ 0 & \frac{c}{\lambda} & \frac{-b}{\lambda} & 0 \\ \frac{a}{|\mathbf{V}|} & \frac{b}{|\mathbf{V}|} & \frac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If both b and c are zero, then $\mathbf{V} = a\mathbf{I}$, and so $\lambda = 0$. In this case, only a $\pm 90°$ rotation about the y axis is required. So if $\lambda = 0$, it follows that

$$A_{\mathbf{v}} = R_{-\theta_2,J} = \begin{pmatrix} 0 & 0 & \frac{-a}{|a|} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{|a|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the same manner we calculate the inverse transformation that aligns the vector $\mathbf{K}$ with the vector $\mathbf{V}$:

$$A_{\mathbf{v}}^{-1} = (R_{-\theta_2,J} \cdot R_{\theta_1,I})^{-1} = R_{\theta_1,I}^{-1} \cdot R_{-\theta_2,J}^{-1} = R_{-\theta_1,I} \cdot R_{\theta_2,J}$$

$$= \begin{pmatrix} \frac{\lambda}{|\mathbf{V}|} & 0 & \frac{a}{|\mathbf{V}|} & 0 \\ \frac{-ab}{\lambda|\mathbf{V}|} & \frac{c}{\lambda} & \frac{b}{|\mathbf{V}|} & 0 \\ \frac{-ac}{\lambda|\mathbf{V}|} & -\frac{b}{\lambda} & \frac{c}{|\mathbf{V}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solved Problems:

Problem 14

Find the transformation for mirror reflection with respect to a given plane. Refer to Fig. 6-9.

SOLUTION

Let the plane of reflection be specified by a normal vector **N** and a reference point $P_0(x_0, y_0, z_0)$. To reduce the reflection to a mirror reflection with respect to the xy plane:

1. Translate P_0 to the origin:
2. Align the normal vector **N** with the vector **K** normal to the xy plane.
3. Perform the mirror reflection in the xy plane (Prob. 6.6).
4. Reverse steps 1 and 2.

So, with translation vector $\mathbf{V} = -x_0\mathbf{I} - y_0\mathbf{J} - z_0\mathbf{K}$

$$M_{\mathbf{N},P_0} = T_{\mathbf{V}}^{-1} \cdot A_{\mathbf{N}}^{-1} \cdot M \cdot A_{\mathbf{N}} \cdot T_{\mathbf{V}}$$

Here, $A_{\mathbf{N}}$ is the alignment matrix defined in Prob. 6.2. So if the vector $\mathbf{N} = n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}$, then from Prob.

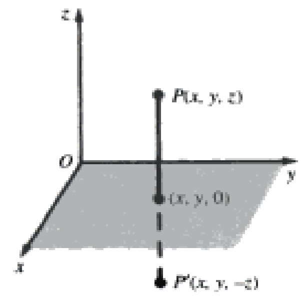


Fig. 6-8

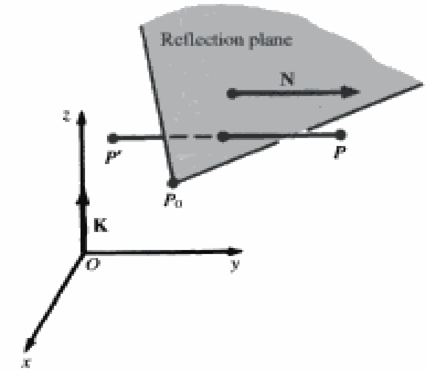


Fig. 6-9

6.2, with $|\mathbf{N}| = \sqrt{n_1^2 + n_2^2 + n_3^2}$ and $\lambda = \sqrt{n_2^2 + n_3^2}$, we find

$$A_{\mathbf{N}} = \begin{pmatrix} \frac{\lambda}{|\mathbf{N}|} & \frac{-n_1 n_2}{\lambda|\mathbf{N}|} & \frac{-n_1 n_3}{\lambda|\mathbf{N}|} & 0 \\ 0 & \frac{n_3}{\lambda} & \frac{-n_2}{\lambda} & 0 \\ \frac{n_1}{|\mathbf{N}|} & \frac{n_2}{|\mathbf{N}|} & \frac{n_3}{|\mathbf{N}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_{\mathbf{N}}^{-1} = \begin{pmatrix} \frac{\lambda}{|\mathbf{N}|} & 0 & \frac{n_1}{|\mathbf{N}|} & 0 \\ \frac{-n_1 n_2}{\lambda|\mathbf{N}|} & \frac{n_3}{\lambda} & \frac{n_2}{|\mathbf{N}|} & 0 \\ \frac{-n_1 n_3}{\lambda|\mathbf{N}|} & \frac{-n_2}{\lambda} & \frac{n_3}{|\mathbf{N}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In addition

$$T_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, from Prob. 6.6, the homogeneous form of M is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solved Problems:

Problem 15

Find the matrix for mirror reflection with respect to the plane passing through the origin and having a normal vector whose direction is $\mathbf{N} = \mathbf{I} + \mathbf{J} + \mathbf{K}$.

SOLUTION

From Prob. 6.7, with $P_0(0,0,0)$ and $\mathbf{N} = \mathbf{I} + \mathbf{J} + \mathbf{K}$, we find $|\mathbf{N}| = \sqrt{3}$ and $\lambda = \sqrt{2}$. Then

$$T_{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\mathbf{V} = 0\mathbf{I} + 0\mathbf{J} + 0\mathbf{K}) \qquad T_{\mathbf{V}}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{\mathbf{N}} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{\mathbf{N}}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The reflection matrix is

$$M_{\mathbf{N},O} = T_{\mathbf{V}}^{-1} \cdot A_{\mathbf{N}}^{-1} \cdot M \cdot A_{\mathbf{N}} \cdot T_{\mathbf{V}}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That's All