Modeling Curved Surfaces

Curved Surfaces

The use of curved surfaces allows for a higher level of modeling, especially for the construction of highly realistic models.

There are several approaches to modeling curved surfaces:

(1) Similar to polyhedral models,
we model an object by using small **curved surface patches** (instead of polygons) placed next to each other.

(2) Another approach is solid modeling,
that constructs a model using elementary solid objects (such as: polyhedra, spheres, cones etc.) as building blocks.

Curved Surfaces

There are two ways to construct a model:

Additive Modeling

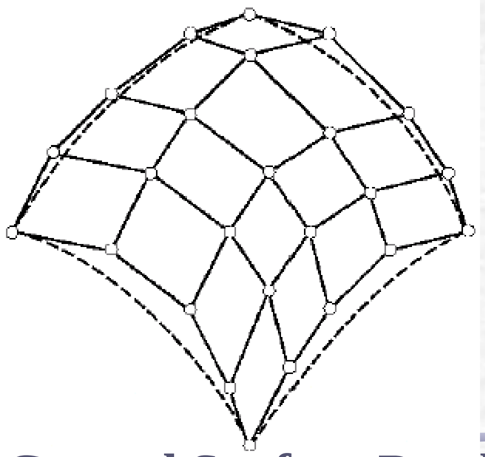
This is the process of building the model by assembling many simpler objects.

Subtractive Modeling

This is the process of removing pieces from a given object to create a new object.

For example, creating a (cylindrical) hole in a sphere or a cube.

We can represent curved surfaces using mesh of curves. So we learn to create curves first then move to curved surfaces.



Curved Surface Patch

Curve Representation

- There are three ways to represent a curve
  - Explicit: $y = f(x)$

    $y = mx + b$ **$y = x^2$**

    (–) Must be a single valued function

    (–) Vertical lines, say $x = d$?
  - Implicit: $f(x,y) = 0$

    $x^2 + y^2 - r^2 = 0$

    (+) **y** can be multiple valued function of x

    (–) Vertical lines?
  - Parametric: $(x, y) = (x(t), y(t))$

    $(x, y) = (\cos t, \sin t)$

    (+) Easy to specify, modify and control

    (–) Extra hidden variable t, the parameter

Explicit Representation

- Curve in 2D: $y = f(x)$
- Curve in 3D: $y = f(x)$, $z = g(x)$
- Surface in 3D: $z = f(x,y)$
- Problems:
  - How about a vertical line $x = c$ as $y = f(x)$?
  - Circle $y = \pm (r^2 - x^2)^{1/2}$ two or zero values for x
- • Rarely used in computer graphics

Implicit Representation

- Curve in 2D: $f(x,y) = 0$
  - Line: $ax + by + c = 0$
  - Circle: $x^2 + y^2 - r^2 = 0$
- Surface in 3d: $f(x,y,z) = 0$
  - Plane: $ax + by + cz + d = 0$
  - Sphere: $x^2 + y^2 + z^2 - r^2 = 0$
- $f(x,y,z)$ can describe 3D object:
  - Inside: $f(x,y,z) < 0$
  - Surface: $f(x,y,z) = 0$
  - Outside: $f(x,y,z) > 0$

Parametric Form for Curves

- Curves: single parameter u (e.g. time)
  - $x = x(t), y = y(t), z = z(t)$
- Circle:
  - $x = \cos(t), y = \sin(t), z = 0$
- Tangent described by derivative

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad \frac{dp(t)}{dt} = \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{bmatrix}$$

- Magnitude is “velocity”

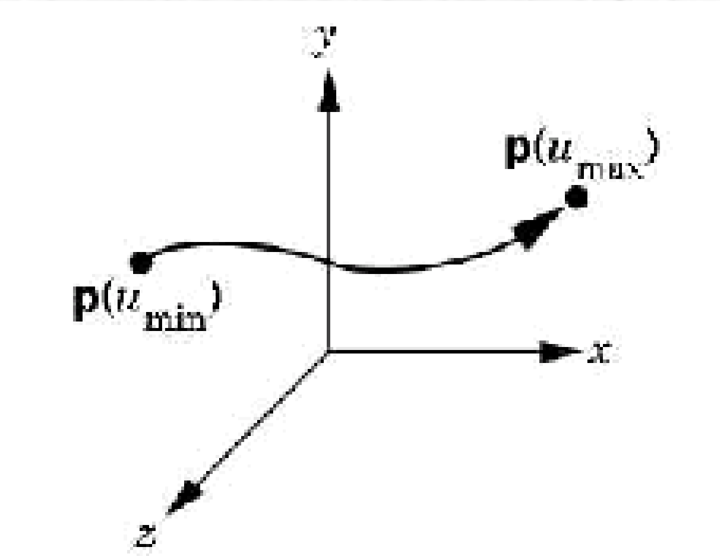
Parametric Form for Surfaces

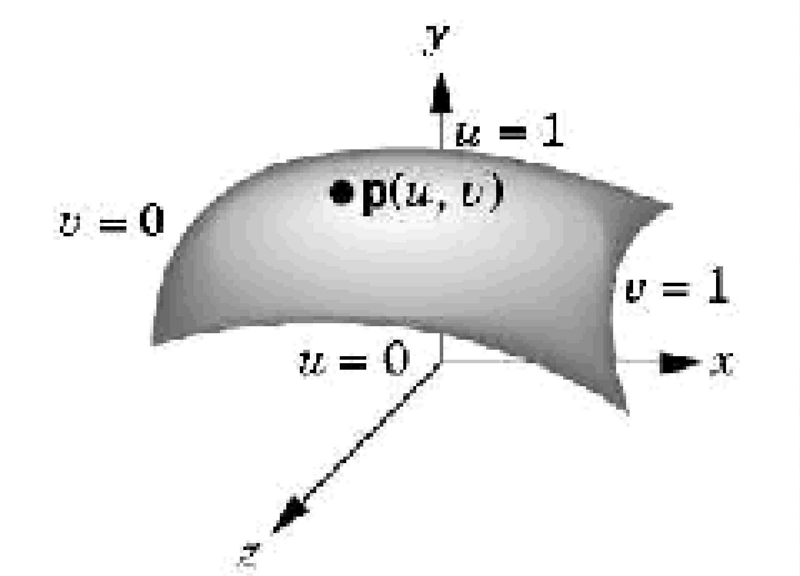
- Use parameters u and v
  - $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$
- Describes surface as both u and v vary
- Partial derivatives describe tangent plane at each point $p(u,v) = [x(u,v)\ y(u,v)\ z(u,v)]^T$

$$\frac{\partial p(u,v)}{\partial u} = \begin{bmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{bmatrix} \qquad \frac{\partial p(u,v)}{\partial v} = \begin{bmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{bmatrix}$$

Advantages of Parametric Form

- Parameters often have natural meaning
- Easy to define and calculate
  - Tangent and normal
  - Curves segments (for example, $0 \leq u \leq 1$)
  - Surface patches (for example, $0 \leq u,v \leq 1$)





Lagrange Polynomial

- Given n+1 points (x_0, y_0), (x_1, y_1) …….. (x_n, y_n)
- To construct a curve that passes through these points we can use Lagrange polynomial defined as follows:.

$$y = f(x) = \sum_{k=0}^{n} y_k L_{n,k}$$

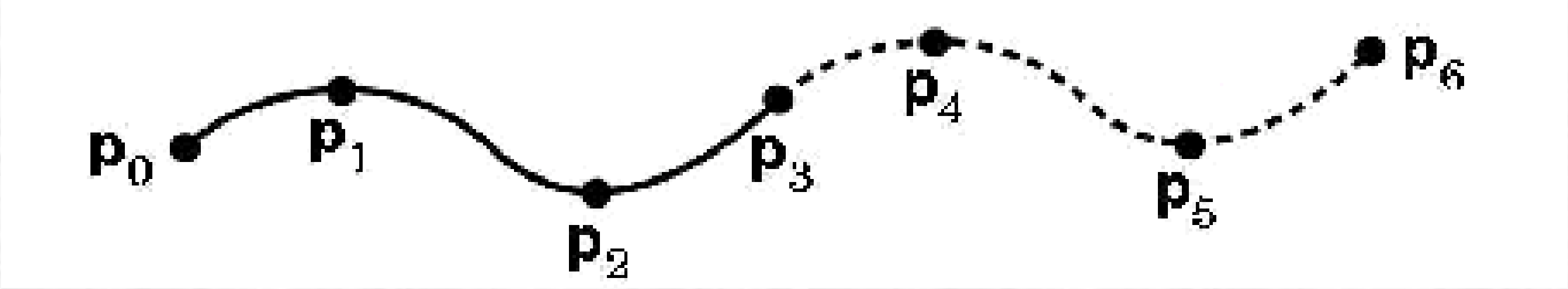
$$L_{n,k} = \frac{(x - x_o)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_o)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

Problems:

- y=f(x), no multiple values
- Higher order functions tend to oscillate
- No local control (change any (x_i, y_i) changes the whole curve)
- Computationally expensive due to high degree.

Piecewise Linear Polynomial

- To overcome the problems with Lagrange polynomial
  - Divide given points into overlap sequences of 4 points
  - construct **3rd degree** polynomial that passes through these points, p_0, p_1 , p_2 , p_3 then p_3, p_4 , p_5 , p_6 etc.
  - Then glue the curves so that they appear **sufficiently smooth** at joint points.



Questions:

1. Why 3rd Degree curves used?
2. How to measure smoothness at joint point?

Why Cubic Curves?

A curve is approximated by a piecewise polynomial curve.

Cubic polynomials are most often used because:

(1) Lower-degree polynomials offer too little flexibility in controlling the shape of the curve.

(2) Higher-degree polynomials can introduce unwanted wiggles and also require more computation.

(3) No lower-degree representation allows a curve segment to be defined by two given endpoints with given derivative at each endpoints.

(4) No lower-degree curves are nonplanar in 3D.

Measure of Smoothness

G^0 Geometric Continuity ⇔ C^0 Parametric Continuity

If two curve segments join together.

G^1 Geometric Continuity

If the **directions** (but not necessarily the magnitudes) of the two segments' tangent vectors are equal at a join point.

C^1 Parametric Continuity

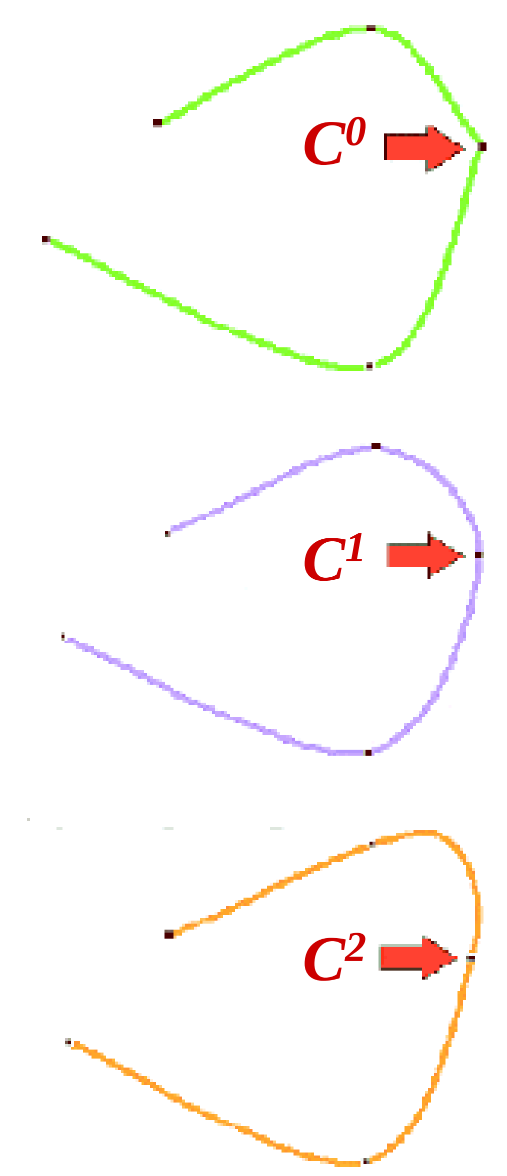
If the **directions and magnitudes** of the two segments' tangent vectors are equal at a join point.

C^2 Parametric Continuity

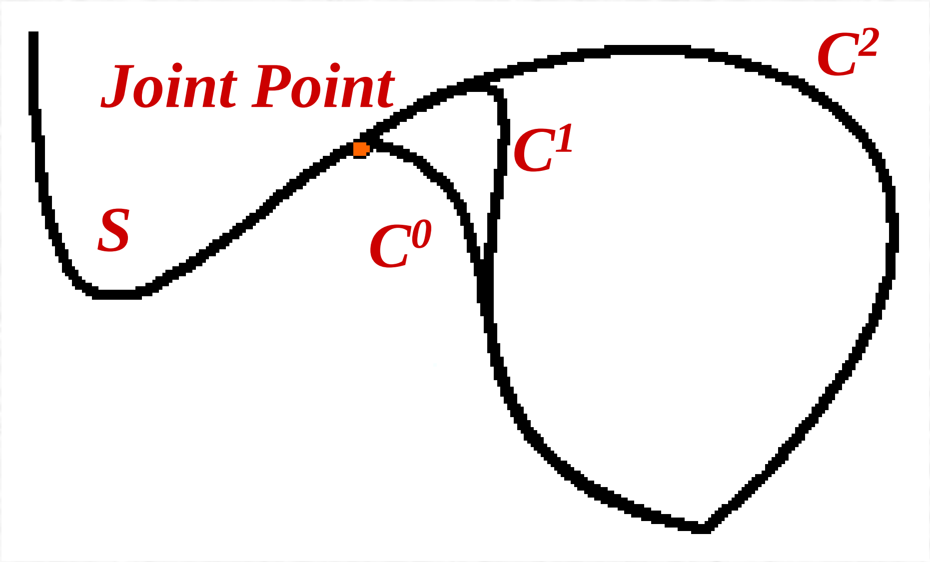
If the direction and magnitude of $Q^2(t)$ (curvature or **acceleration**) are equal at the join point.

C^n Parametric Continuity

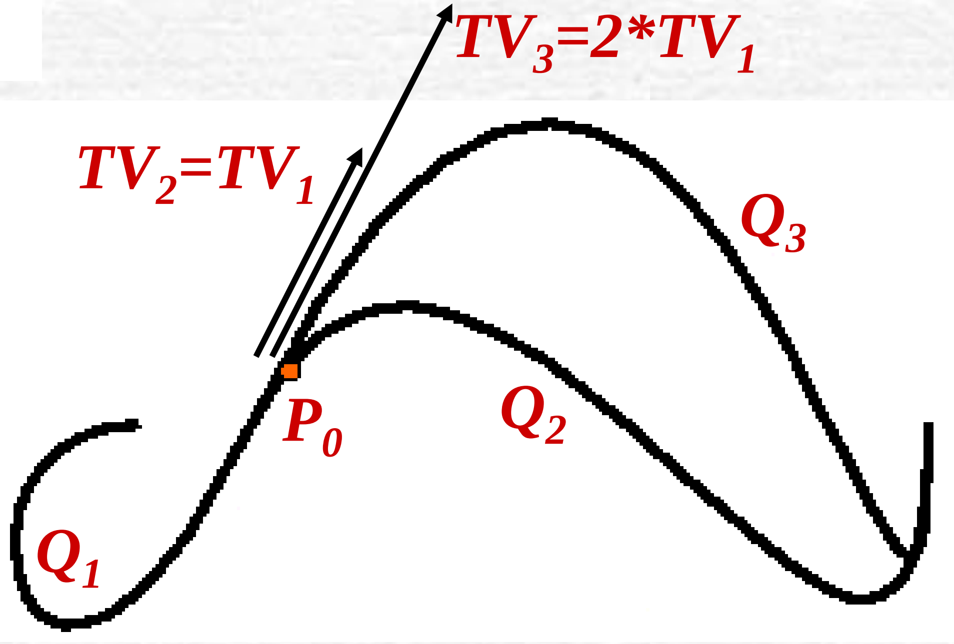
If the direction and magnitude of $Q^n(t)$ through the nth derivative are equal at the join point.



Measure of Smoothness



- By increasing parametric continuity we can increase smoothness of the curve.



- Q_1 & Q_2 are C^1 and G^1 continuous
- Q_1 & Q_3 are G^1 continuous only as Tangent vectors have different magnitude.
- Observe the effect of increasing in magnitude of TV

Desirable Properties of a Curve

- Simple control
  - lines need only two points
  - curves will need more (but not significantly more)
- Intuitive control
  - Physically meaningful quantities like position, tangent, curvature etc.
- Global Vs. Local Control
  - Portion of curve effected by a control point.
- General Parameterization
  - Handle multi-valued x-y mapping

Desirable Properties of a Curve cont'd

- Interpolation Vs Approximation
- Axis Independent
  - Equation might change but the shape remain same under a coordinate transform
  - (translation, rotation, scaling) of a curve = (translation, rotation, scaling) of its control points.
- Degree of Smoothness.
  - May need more or less
  - May need varying degrees of smoothness in a single curve.

Interpolation Vs. Approximation

Given $n + 1$ points $P_0(x_0, y_0), P_1(x_1, y_1), \ldots, P_n(x_n, y_n)$

we wish to find a curve that, in some sense, fits the shape outlined by these points.

Based on requirements we are faced with two problems:

Interpolation

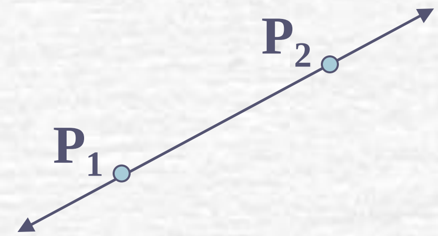
If we require the curve to pass through all the points.

Approximation

If we require only that the curve be near these points.

Parametric Representation of Lines

- Interpolation of two points
- In Parametric form:



$$P(t) = P_1 + t \cdot (P_2 - P_1)$$

$$x(t) = x_1 + t \cdot (x_2 - x_1)$$

$$y(t) = y_1 + t \cdot (y_2 - y_1)$$

$$x(t) = TC_x = TMG_x = BG_x$$

$$y(t) = TC_y = TMG_y = BG_y$$

$$x(t) = \underbrace{\begin{bmatrix} t & 1 \end{bmatrix}}_{\text{Parameter } T} \underbrace{\begin{bmatrix} x_2 - x_1 \\ x_1 \end{bmatrix}}_{\text{Co-eff } C} = \begin{bmatrix} t & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{Basis Matrix } M} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Geometry } G} = \underbrace{\begin{bmatrix} 1-t & t \end{bmatrix}}_{\text{Blending Function } B} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Parameter **T**

Co-eff **C**

Basis Matrix **M**

Geometry **G**

Blending Function **B**

Parametric Cubic Curves

$$Q(t) = [x(t)\ y(t)\ z(t)] \begin{cases} x(t) = a_x t^3 + b_x t^2 + c_x t + d_x, \\ y(t) = a_y t^3 + b_y t^2 + c_y t + d_y, \\ z(t) = a_z t^3 + b_z t^2 + c_z t + d_z,\ 0 \le t \le 1 \end{cases}$$

$$\therefore Q(t) = \underbrace{\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}}_{C}$$

$$\therefore Q(t) = T \cdot C$$

Parametric Cubic Curves

- Now co-efficient matrix **C** can be expressed as a multiple of basis(weight) matrix **M** and geometry matrix **G**.

$$Q(t) = [x(t) \;\; y(t) \;\; z(t)] = T \cdot C = T \cdot M \cdot G$$

$$Q(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \underbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}}_{\text{basis matrix}} \underbrace{\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}}_{\text{geometry vector}}$$

- Each element of geometry vector **G** has 3 component for x, y and z.
- Components of **G** can be expressed as G_x, G_y and G_z.

Parametric Cubic Curves

- Multiplying out only the x-component we get

$$x(t) = T \cdot M \cdot G_x = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} g_{1x} \\ g_{2x} \\ g_{3x} \\ g_{4x} \end{bmatrix}$$

$$\begin{aligned} x(t) = {} & \left(t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}\right) g_{1x} \\ & + \left(t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}\right) g_{2x} \\ & + \left(t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}\right) g_{3x} \\ & + \left(t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}\right) g_{4x} \end{aligned}$$

a blending function

- The curve is a weighted sum of the elements of geometry matrix
- The weights are each cubic polynomials of t called *blending function*

Derivative of *Q(t)*

- Derivative of ***Q(t)*** is the parametric *tangent vector* of the curve.

$$\frac{dQ(t)}{dt} = Q'(t) = \left[\frac{d}{dt}x(t) \quad \frac{d}{dt}y(t) \quad \frac{d}{dt}z(t) \right]$$

$$Q'(t) = \frac{d}{dt}T \cdot C = \left[3t^2 \quad 2t \quad 1 \quad 0\right] \cdot C$$

$$Q'(t) = \left[3a_x t^2 + 2b_x t + c_x \qquad 3a_y t^2 + 2b_y t + c_y \qquad 3a_z t^2 + 2b_z t + c_z\right]$$

Curve Design : Determining C

A curve segment $Q(t)$ is defined by constraints on:

(1) endpoints

(2) tangent vectors

and (3) continuity between segments

Each cubic polynomial of $Q(t)$ has 4 coefficients,
so 4 constraints will be needed,
allowing us
to formulate 4 equations in the 4 unknowns,
then solving for the unknowns.

Hermit Curves

A cubic Hermite curve segment interpolating the endpoints P_1 and P_4 is determined by constraints on
the endpoints P_1 and P_4
and
tangent vectors at the endpoints R_1 and R_4

Hermit Curves

The Hermite Geometry Vector: $G_H = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}$

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x = T \cdot C_x = T \cdot M_H \cdot G_{H_x}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_H \cdot G_{H_x}$$

The constraints on $x(0)$ and $x(1)$:

$$x(0) = P_{1x} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} M_H \cdot G_{H_x}$$

$$x(1) = P_{4x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} M_H \cdot G_{H_x}$$

Hermit Curves

$$x'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} M_H \cdot G_{H_x}$$

Hence the tangent-vector constraints:

$$x'(0) = R_{1x} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} M_H \cdot G_{H_x}$$
$$x'(1) = R_{4x} = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} M_H \cdot G_{H_x}$$

The 4 constraints can be written as:

$$\begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}_x = G_{H_x} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} M_H \cdot G_{H_x}$$

Hermit Curves

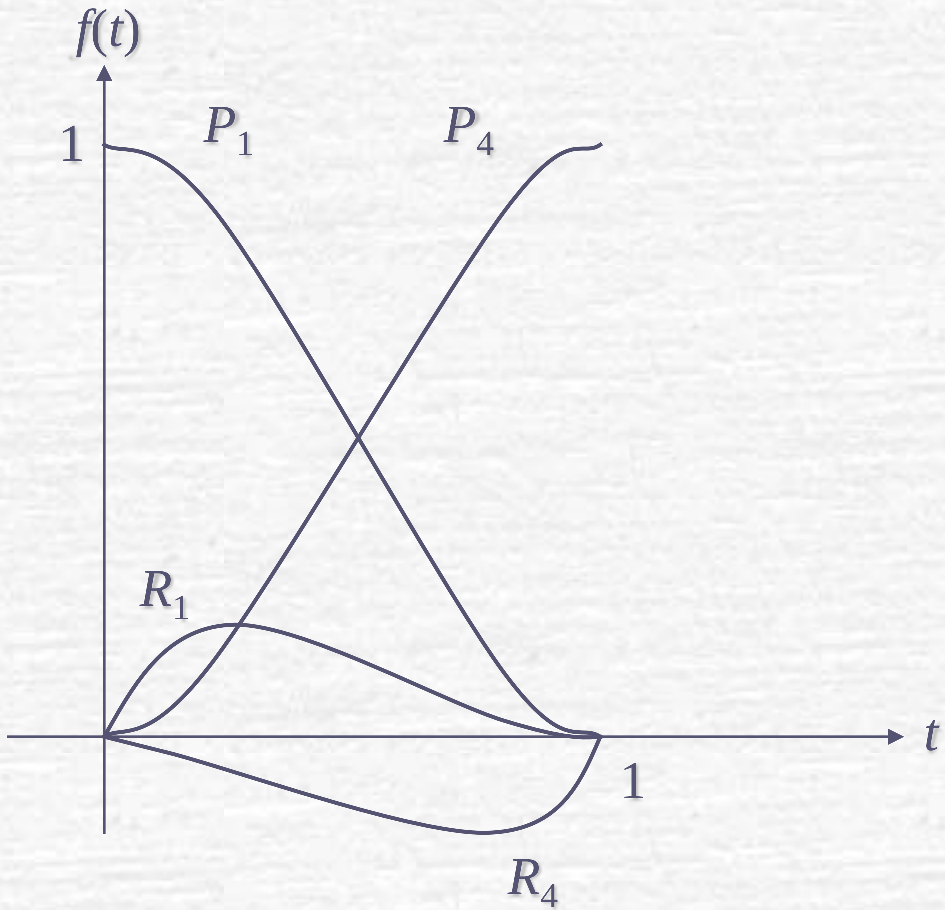
$$M_H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q(t) = [x(t) \quad y(t) \quad z(t)] = T \cdot M_H \cdot G_H = B_H \cdot G_H$$

$$= (2t^3 - 3t^2 + 1)P_1 + (-2t^3 + 3t^2)P_4$$

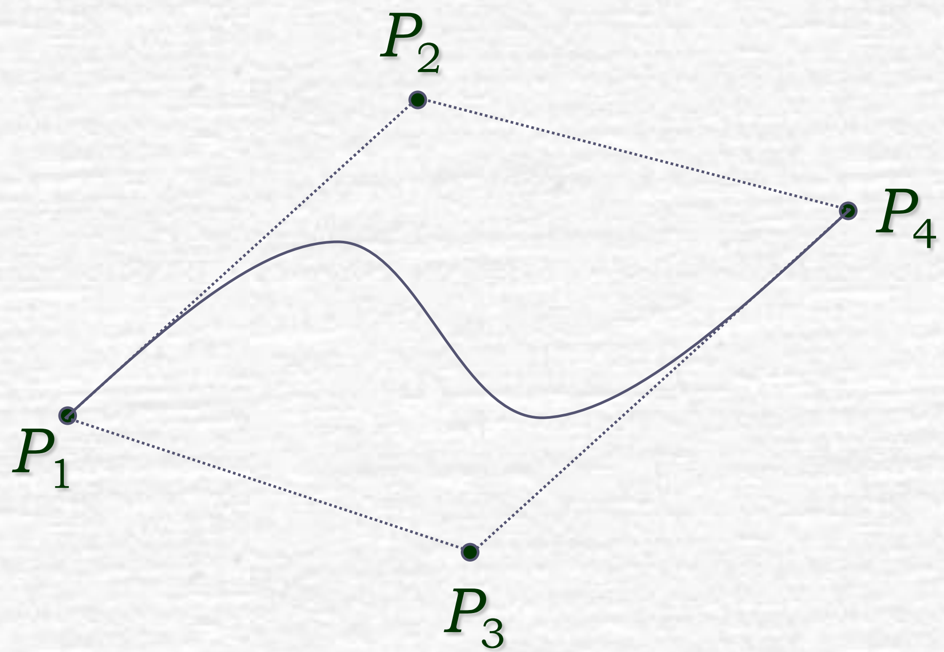
$$+ (t^3 - 2t^2 + t)R_1 + (t^3 - t^2)R_4$$

Hermit Curves



The Hermite Blending Functions

Bézier Curves



Indirectly specifies the endpoint tangent vectors by specifying two intermediate points that are not on the curve.

$$R_1 = Q'(0) = P_1P_2 = 3(P_2 - P_1)$$
$$R_4 = Q'(1) = P_3P_4 = 3(P_4 - P_3)$$

Bézier Curves

The Bézier Geometry Vector: $G_B = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$

$$G_H = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = M_{HB} \cdot G_B$$

$$\begin{aligned} Q(t) &= T \cdot M_H \cdot G_H = T \cdot M_H \cdot (M_{HB} \cdot G_B) \\ &= T \cdot (M_H \cdot M_{HB}) \cdot G_B = T \cdot M_B \cdot G_B \end{aligned}$$

Bézier Curves

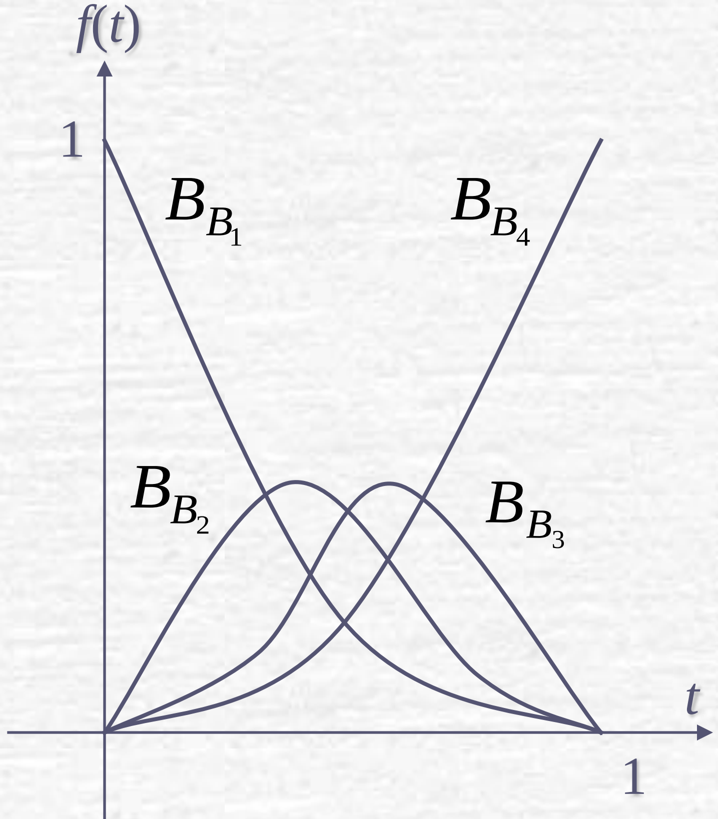
$$M_B = M_H \cdot M_{HB} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q(t) = T \cdot M_B \cdot G_B$$

$$= (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t)P_3 + t^3 P_4$$

The 4 polynomials in $B_B = T \,.\, M_B$ are called the Bernstein polynomials.

Bézier Curves

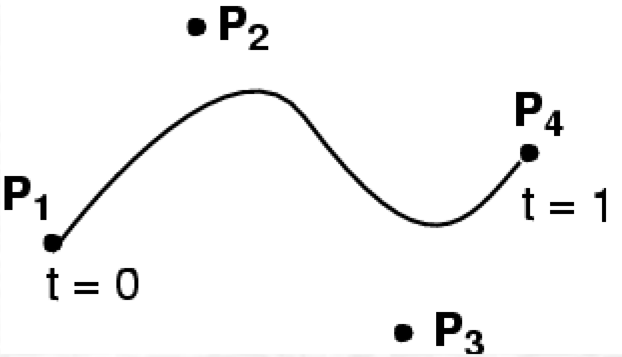


The Bernstein Polynomials

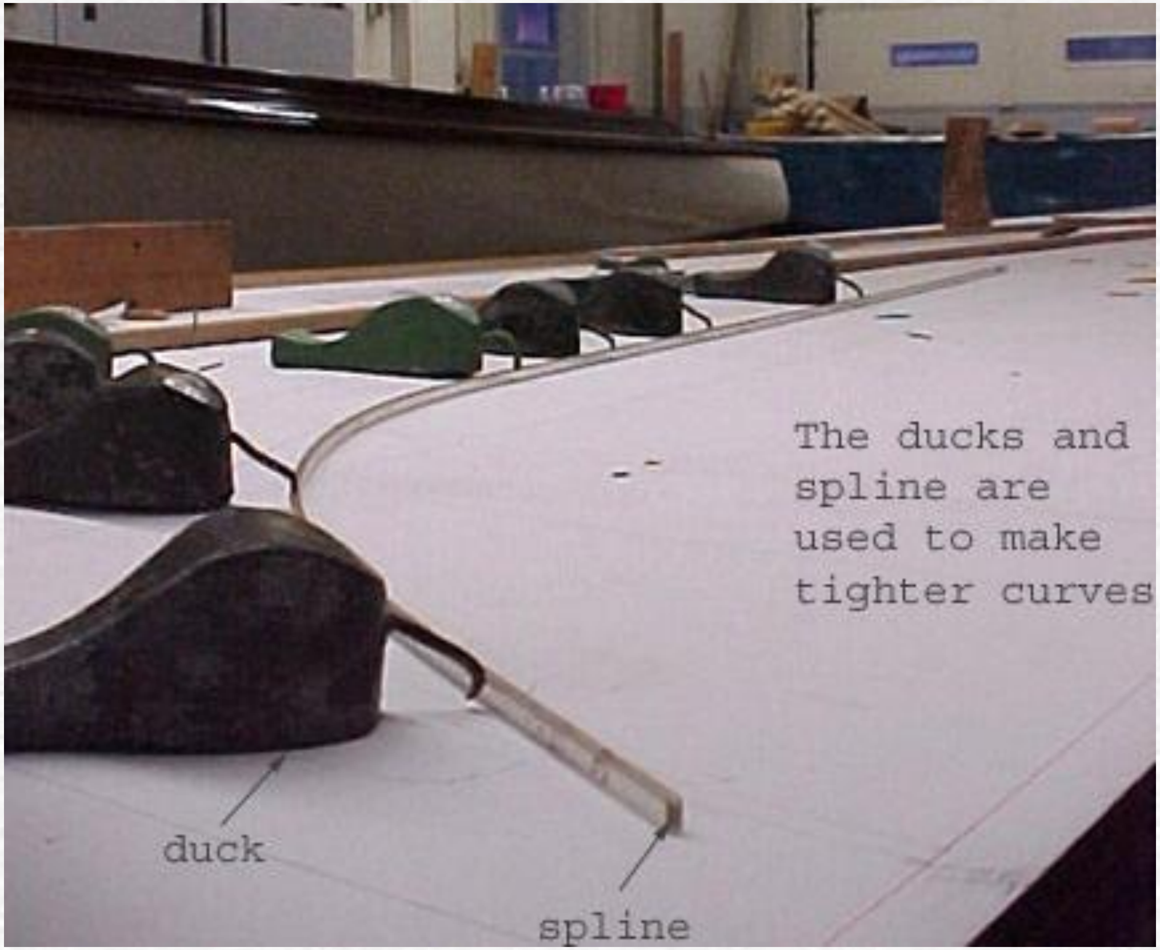
A Bézier curve is bounded by the convex hull of its control points.

What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve



Splines

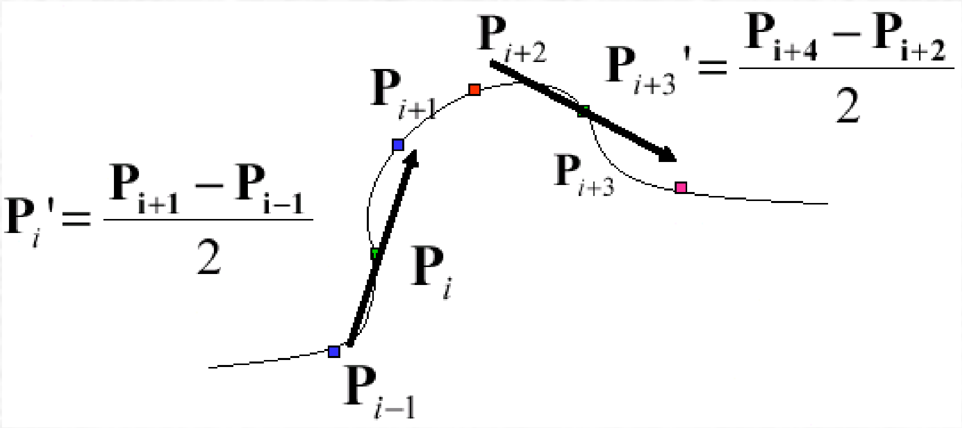

The ducks and
spline are
used to make
tighter curves
duck
spline

Other Curves

- Natural Cubic Spline
  - C^0, C^1 and C^2 continuous cubic polynomial
  - Smoother than previous curves which don't have C^2 continuity.
  - Interpolates all of the control points
- B-Splines
  - C^0, C^1 and C^2 continuous cubic polynomial
  - Don't interpolate the control points
  - Varieties:
    - Uniform Vs Nonuniform
    - Rational Vs Non-rational
- Catmull-Rom splines
- And many more….

Catmull-Rom Splines

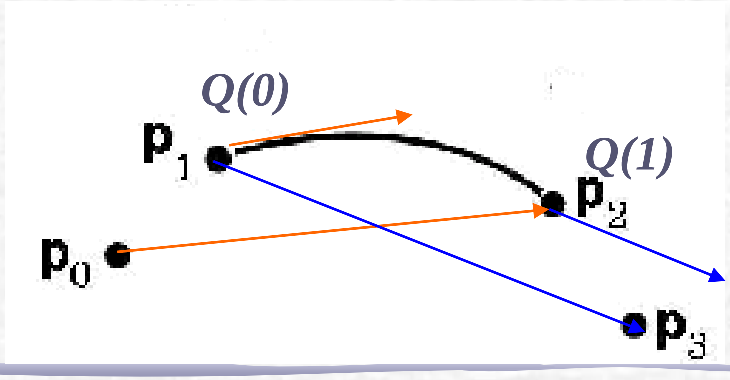
- Interpolation splines i.e. Passes through all control points.
- Tangent at any control point is parallel to the line joining the control points adjacent to that point.



$$Q(t) = T \cdot M_{CR} \cdot G_{Bs_i}$$

$$= T \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{i-2} \\ P_{i-1} \\ (P_{i-1} - P_{i-3})/2 \\ (P_i - P_{i-2})/2 \end{bmatrix}$$

Hermite Basis



$$= \frac{1}{2} \cdot T \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

Curve Rendering

- Brute-Force method
- Forward differencing
- Recursive sub-division

Brute-Force method

```
t = 0;
for (i=0; i <= 100; i++) {
        x(t) = a_x t^3 + b_x t^2 + c_x t + d_x
        y(t) = a_y t^3 + b_y t^2 + c_y t + d_y
        x(t) = a_z t^3 + b_z t^2 + c_z t + d_z
        Plot3d( x(t), y(t), z(t) );
        t += 0.01;
}
```

Curve Rendering

Forward differencing method

$$f(t) = at^3 + bt^2 + ct + d$$

$$f(t+\delta) = a(t+\delta)^3 + b(t+\delta)^2 + c(t+\delta) + d$$

$$\Delta f(t) = f(t+\delta) - f(t)$$

$$= 3a\delta t^2 + (3a\delta^2 + 2b\delta)t + (a\delta^3 + b\delta^2 + c\delta)$$

$$\underline{f(t+\delta) = f(t) + \Delta f(t)}$$

$$\Delta f(t) = 3a\delta t^2 + (3a\delta^2 + 2b\delta)t + (a\delta^3 + b\delta^2 + c\delta)$$

$$\Delta f(t+\delta) = 3a\delta(t+\delta)^2 + (3a\delta^2 + 2b\delta)(t+\delta) + (a\delta^3 + b\delta^2 + c\delta)$$

$$\Delta^2 f(t) = \Delta f(t+\delta) - \Delta f(t)$$

$$= 6a\delta^2 t + (6a\delta^3 + 2b\delta^2)$$

$$\underline{\Delta f(t+\delta) = \Delta f(t) + \Delta^2 f(t)}$$

Curve Rendering

Forward differencing method

$$\Delta^2 f(t) = 6a\delta^2 t + (6a\delta^3 + 2b\delta^2)$$

$$\Delta^2 f(t+\delta) = 6a\delta^2 (t+\delta) + (6a\delta^3 + 2b\delta^2)$$

$$\Delta^3 f(t) = \Delta^2 f(t+\delta) - \Delta^2 f(t)$$

$$= 6a\delta^3$$

$$\underline{\Delta^2 f(t+\delta) = \Delta^2 f(t) + \Delta^3 f(t)}$$

$$f_o = d$$

$$\Delta f_o = a\delta^3 + b\delta^2 + c\delta$$

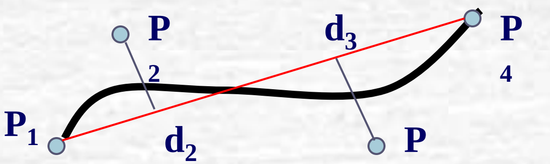
$$\Delta^2 f_o = 6a\delta^3 + 2b\delta^2$$

$$\Delta^3 f_o = 6a\delta^3$$

```
δ = 1 / n;
f =d;   Δf = aδ³+ bδ³+ cδ;   Δf ²= 6aδ³+ 2bδ²;   Δf ³= 6aδ³;
Plot(f.x, f.y, f.z );
for (i=0; i <= n; i++) {
        f += Δf ;      Δf += Δf ²;      Δf ²+= Δf ³;
         Plot(f.x, f.y, f.z );
}
```

Curve Rendering

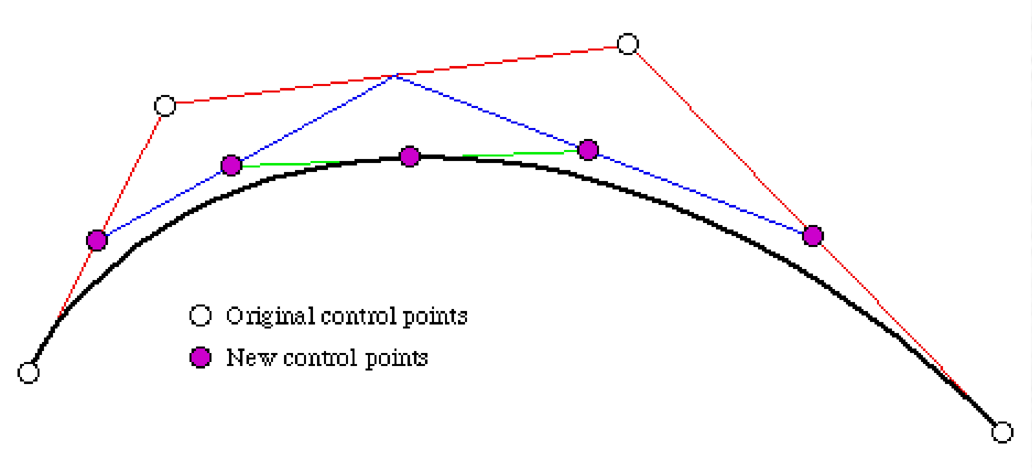
Recursive sub-division



Straight if $d_2 < \varepsilon$ and $d_3 < \varepsilon$

```
Void DrawCurveRecSub(curve, ε )
{
    if (Straight(curve, ε )
        DrawLine(curve);
    else {
        SubdivideCurve(curve, leftCurve, rightCurve);
        DrawCurveRecSub(leftCurve, ε );
            DrawCurveRecSub(RightCurve, ε );
    }
}
```

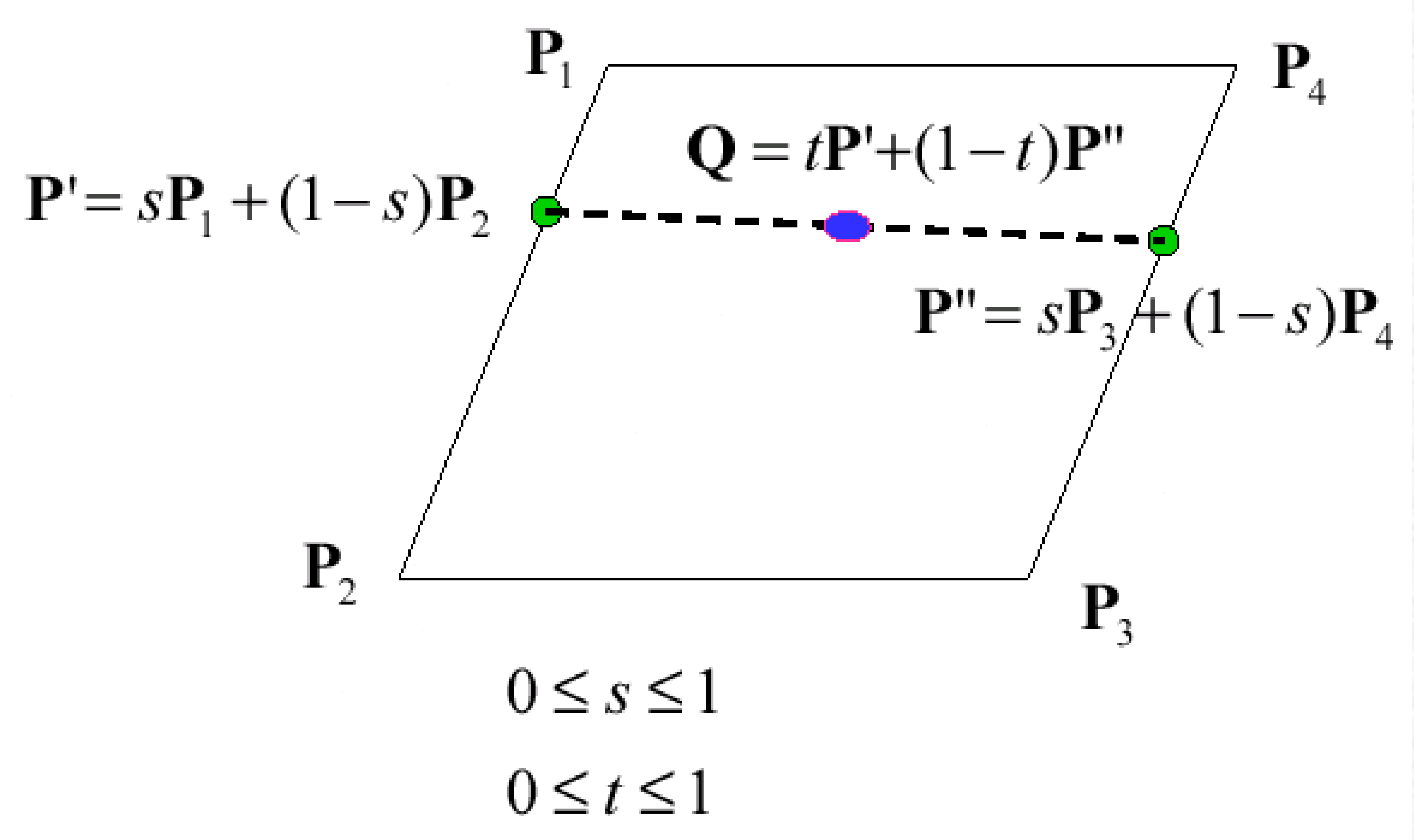
Sub-Division of Bézier Curves



Curved Surfaces

Planer Surfaces

- Using bi-linear interpolation



Parametric Bicubic Surfaces

Curved surface using cartesian product

$$Q(t) = at^3 + bt^2 + ct + d$$

$$Q(s,t) = \left(as^3 + bs^2 + cs + d\right)\left(a't^3 + b't^2 + c't + d'\right)$$

$$= a_{33}s^3t^3 + a_{32}s^3t^2 + \Lambda\ \Lambda\ \Lambda + a_{10}s + a_{01}t + a_{00}$$

$$= \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} a' & b' & c' & d' \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \begin{bmatrix} c_{33} & c_{32} & c_{31} & c_{30} \\ c_{23} & c_{22} & c_{21} & c_{20} \\ c_{13} & c_{12} & c_{11} & c_{10} \\ c_{03} & c_{02} & c_{01} & c_{00} \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$\therefore Q(s,t) = S \cdot C \cdot T^T$$

$$C = M_s \cdot G \cdot M_t$$

Parametric Bicubic Surfaces

- Generalization of parametric cubic curves
- The elements of geometry matrix are curves themselves instead of constants.

$$Q(s,t) = S \cdot M_s \cdot G(t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \cdot M_s \cdot \begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \end{bmatrix}$$

$$G_i(t) = T \cdot M_t \cdot \begin{bmatrix} g_{i1} \\ g_{i2} \\ g_{i3} \\ g_{i4} \end{bmatrix}$$

Parametric Bicubic Surfaces

$$\therefore \left[G_1(t) \quad G_2(t) \quad G_3(t) \quad G_4(t)\right] = T \cdot M_t \cdot \begin{bmatrix} g_{11} & g_{21} & g_{31} & g_{41} \\ g_{12} & g_{22} & g_{32} & g_{42} \\ g_{13} & g_{23} & g_{33} & g_{43} \\ g_{14} & g_{24} & g_{34} & g_{44} \end{bmatrix}$$

$$\therefore \begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \cdot M_t^T \cdot T^T$$

$$\Theta \left(A \cdot B \cdot C\right)^T = C^T \cdot B^T \cdot A^T$$

Parametric Bicubic Surfaces

$$Q(s,t) = S \cdot M_s \cdot \begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \end{bmatrix} = S \cdot M_s \cdot \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \cdot M_t^T \cdot T^T$$

$$\therefore Q(s,t) = S \cdot M_s \cdot G \cdot M_t^T \cdot T^T \qquad 0 \le s,t \le 1$$

Written separately for each of

x, y and z, the form is

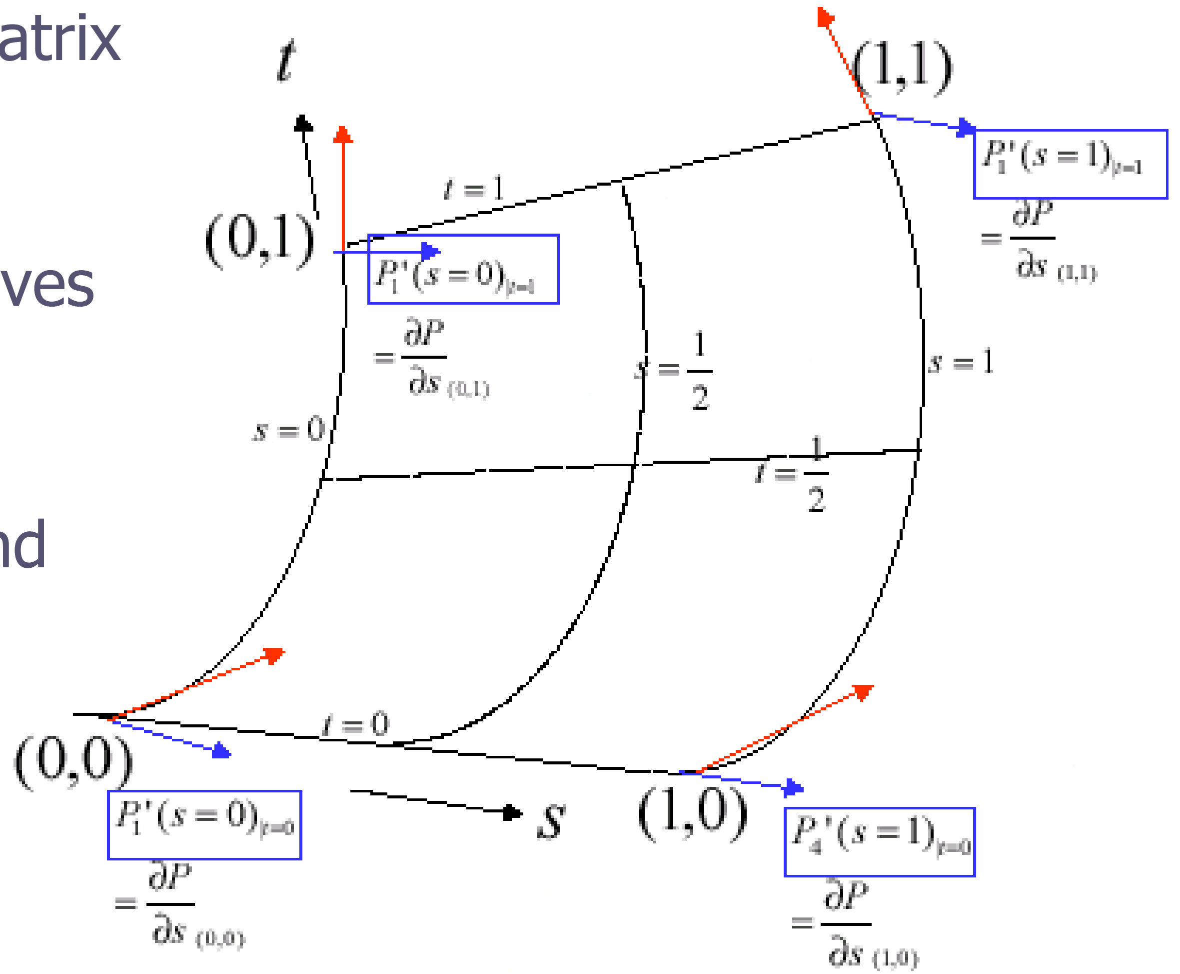
$$x(s,t) = S \cdot M_s \cdot G_x \cdot M_t^T \cdot T$$

$$y(s,t) = S \cdot M_s \cdot G_y \cdot M_t^T \cdot T$$

$$z(s,t) = S \cdot M_s \cdot G_z \cdot M_t^T \cdot T$$

Hermite Surfaces

- Harmite Surfaces are completely defined by a 4X4 geometry Matrix $\mathbf{G_H}$
- The surface is a stacking of (s) Curves
- Each (s) curve is a Harmite Curve
- With end points and end tangents as function of (t)



Hermite Surface

$$P(s) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_4(t) \\ P_1'(t) \\ P_4'(t) \end{bmatrix}$$

$$P_1(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_h \begin{bmatrix} P_{0,0} \\ P_{0,1} \\ \frac{\partial P}{\partial t}_{0,0} \\ \frac{\partial P}{\partial t}_{0,1} \end{bmatrix} \qquad P_1'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_h \begin{bmatrix} \frac{\partial P}{\partial s}_{0,0} \\ \frac{\partial P}{\partial s}_{0,1} \\ \frac{\partial^2 P}{\partial s \partial t}_{0,0} \\ \frac{\partial^2 P}{\partial s \partial t}_{0,1} \end{bmatrix}$$

$$P_4(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_h \begin{bmatrix} P_{1,0} \\ P_{1,1} \\ \frac{\partial P}{\partial t}_{1,0} \\ \frac{\partial P}{\partial t}_{1,1} \end{bmatrix} \qquad P_4'(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} M_h \begin{bmatrix} \frac{\partial P}{\partial s}_{1,0} \\ \frac{\partial P}{\partial s}_{1,1} \\ \frac{\partial^2 P}{\partial s \partial t}_{1,0} \\ \frac{\partial^2 P}{\partial s \partial t}_{1,1} \end{bmatrix}$$

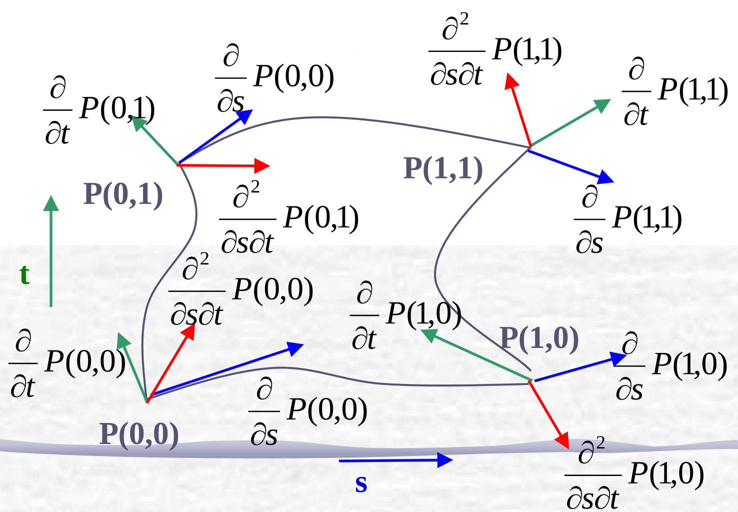
Hermite Surface

$$[P_1(t) \quad P_4(t) \quad P_1'(t) \quad P_4'(t)] = [t^3 \quad t^2 \quad t \quad 1] M_h \begin{bmatrix} P_{0,0} & P_{1,0} & \frac{\partial P}{\partial s}_{0,0} & \frac{\partial P}{\partial s}_{1,0} \\ P_{0,1} & P_{1,1} & \frac{\partial P}{\partial s}_{0,1} & \frac{\partial P}{\partial s}_{1,1} \\ \frac{\partial P}{\partial t}_{0,0} & \frac{\partial P}{\partial t}_{1,0} & \frac{\partial^2 P}{\partial s \partial t}_{0,0} & \frac{\partial^2 P}{\partial s \partial t}_{1,0} \\ \frac{\partial P}{\partial t}_{0,1} & \frac{\partial P}{\partial t}_{1,1} & \frac{\partial^2 P}{\partial s \partial t}_{0,1} & \frac{\partial^2 P}{\partial s \partial t}_{1,0} \end{bmatrix}$$

positions — end tangents

$$P(s,t) = [s^3 \quad s^2 \quad s \quad 1] M_h \begin{bmatrix} P_{0,0} & P_{1,0} & \frac{\partial P}{\partial s}_{0,0} & \frac{\partial P}{\partial s}_{1,0} \\ P_{0,1} & P_{1,1} & \frac{\partial P}{\partial s}_{0,1} & \frac{\partial P}{\partial s}_{1,1} \\ \frac{\partial P}{\partial t}_{0,0} & \frac{\partial P}{\partial t}_{1,0} & \frac{\partial^2 P}{\partial s \partial t}_{0,0} & \frac{\partial^2 P}{\partial s \partial t}_{1,0} \\ \frac{\partial P}{\partial t}_{0,1} & \frac{\partial P}{\partial t}_{1,1} & \frac{\partial^2 P}{\partial s \partial t}_{0,1} & \frac{\partial^2 P}{\partial s \partial t}_{1,0} \end{bmatrix}^T M_h^T \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

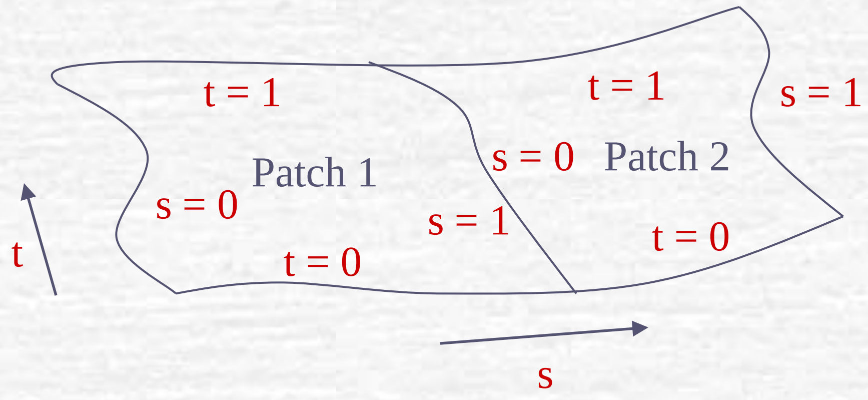
end tangents — twist vectors



Connecting Hermite Patches

To join two patches in **s** direction:

$$\underset{Patch\,1}{\begin{bmatrix} - & - & - & - \\ g_{21} & g_{22} & g_{23} & g_{24} \\ - & - & - & - \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}} \qquad \underset{Patch\,2}{\begin{bmatrix} g_{21} & g_{22} & g_{23} & g_{24} \\ - & - & - & - \\ kg_{41} & kg_{42} & kg_{43} & kg_{44} \\ - & - & - & - \end{bmatrix}}$$



For C^1 continuity k = 1

Bézier Surface

- Bézier geometry matrix **G** consists of 16 control points.
- Bézier bicubic formulation can be derived in exactly the same way as the Hermite cubic:

$$x(s,t) = S \cdot M_B \cdot G_{B_x} \cdot M_B^T \cdot T^T$$

$$y(s,t) = S \cdot M_B \cdot G_{B_y} \cdot M_B^T \cdot T^T$$

$$z(s,t) = S \cdot M_B \cdot G_{B_z} \cdot M_B^T \cdot T^T$$