CURVE

REPRESENTATION

Representation

Non-parametric form: $y = f(X)$

Explicit form: $y = mx + b$

Implicit form: $f(x, y) = 0$

Parametric form:
$x = x(t)$
$y = y(t)$

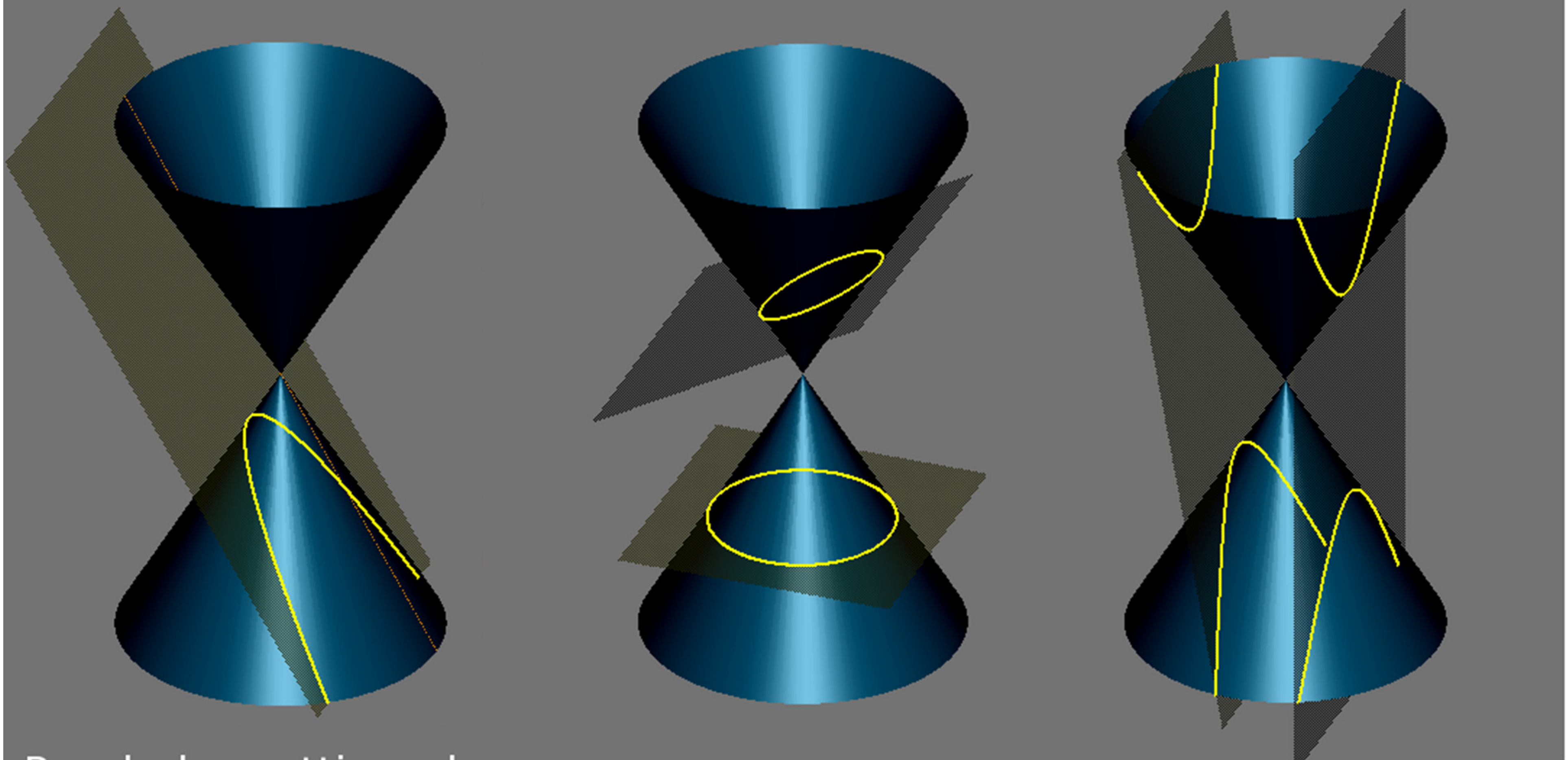
2^{nd} degree implicit representation:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

Any guess, why the factor 2 is used ?

This form of the expression, with the coefficients, provide a wide variety of 2D curve forms called:

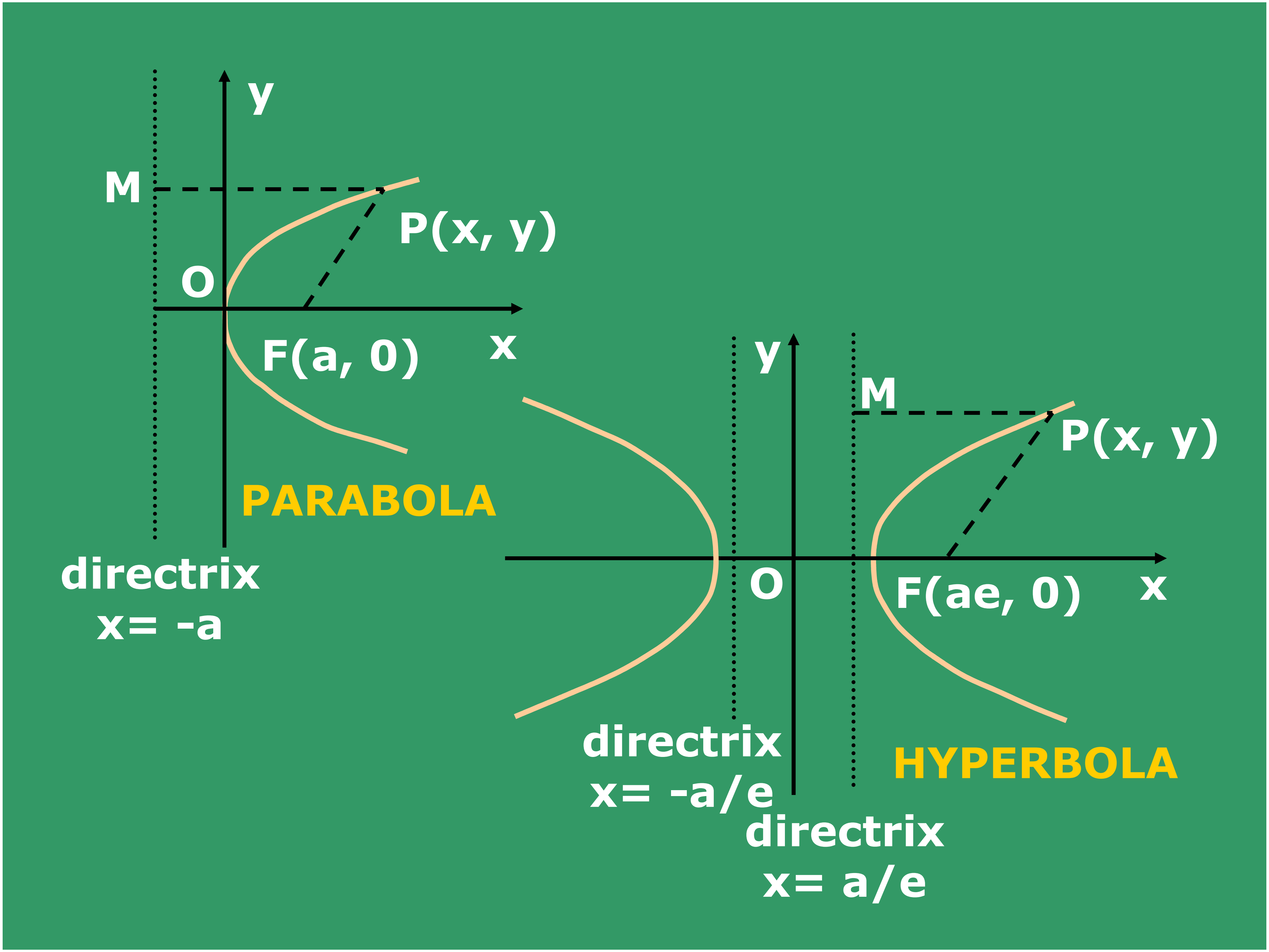
CONIC SECTIONS



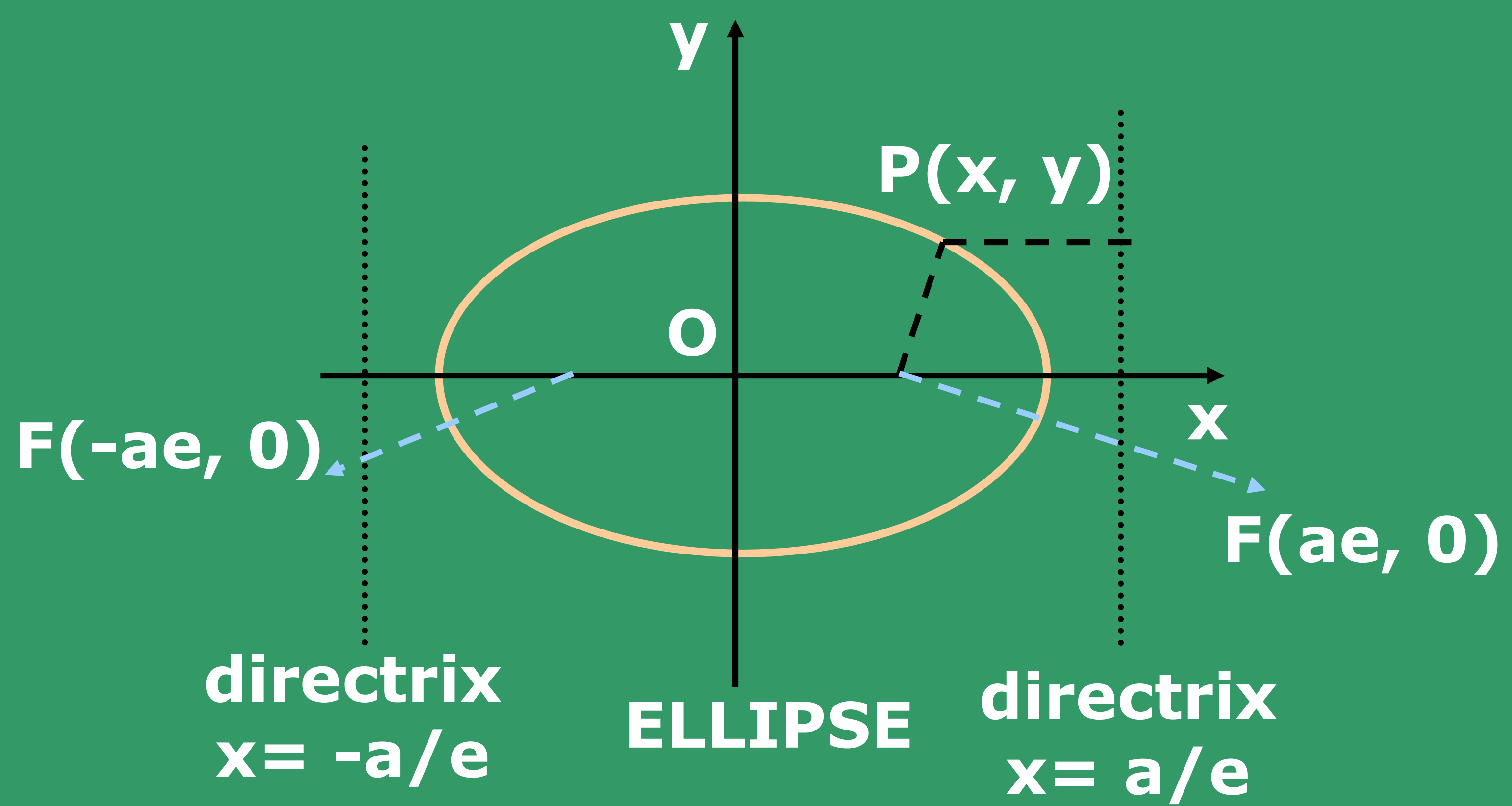
Parabola- cutting plane parallel to side of cone.

Circle and Ellipse

Hyperbolas



y
M
P(x, y)
O
F(a, 0)
x
PARABOLA
directrix
x= -a
y
M
P(x, y)
O
F(ae, 0)
x
directrix
x= -a/e
HYPERBOLA
directrix
x= a/e



y
P(x, y)
O
x
F(-ae, 0)
F(ae, 0)
directrix
x= -a/e
ELLIPSE
directrix
x= a/e

CONIC SECTIONS

PARABOLA

$y^2 = 4ax; a > 0$

$Focus : (a, 0);$

$Directrix = -a.$

$\text{eccentricity}, e = 1$

$x = at^2; y = \pm 2at.$

or

$x = \tan^2(\phi);$

$y = \pm 2\sqrt{a\tan(\phi)}.$

HYPERBOLA

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$

$b^2 = a^2(e^2 - 1);$

$e > 1; Foci : (\pm ae, 0).$

$Directrices : x = \pm a / e;$

$x = a\sec(t),$

$y = b\tan(t);$

$-\pi / 2 < t < \pi / 2.$

Rectangular

Hyperbola :

$\mathrm{e} = \sqrt{2}; x = ct; y = c / t.$

ELLIPSE

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$

$a \geq b > 0.$

$b^2 = a^2(1 - e^2);$

$0 \leq e \leq 1.$

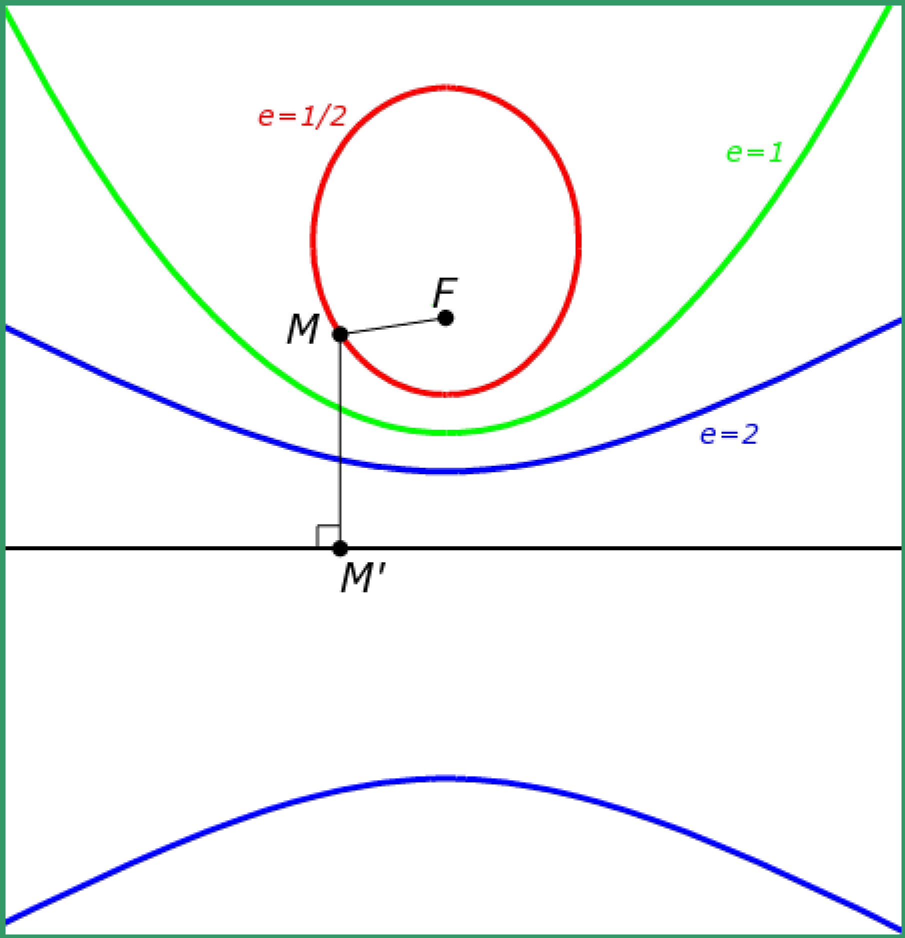
$Foci : (\pm ae, 0);$

$Directrices : x = \pm a / e.$

$x = a\cos(t),$

$y = b\sin(t);$

$t \in [-\pi, \pi].$



Ellipse (e=1/2), parabola (e=1) and hyperbola (e=2) with fixed focus F and directrix.

For circle, e = 0.

Polar Equation of a conic (***<u>home assignment</u>***):

$$r = \frac{e.L}{1 + e\cos(\theta)}, \quad \text{where}, L = dist(F, d)$$

F – Focal Point; d – Directrix;

e – Eccentricity.

Condns: Focal point at Origin;

e.L = l; is called the “semi-latus rectum”.

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

If the conic passes through the origin: f = 0.

Assuming, one of the parameters to be a constant, c = 1.0, f = 1.0

Remaining 5 Coeffs. may be obtained using 5 geometric conditions:

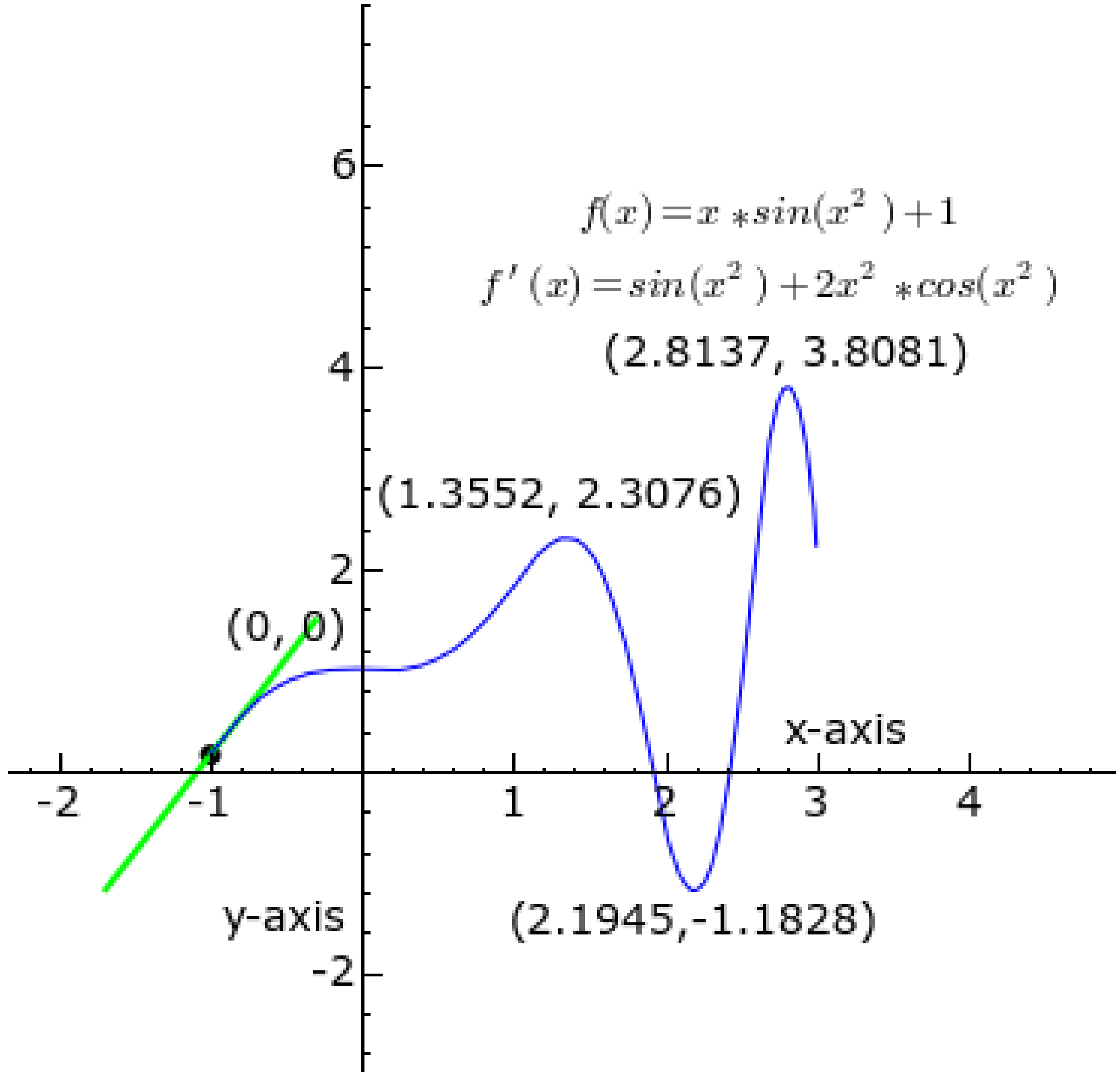
Say:

Boundary Conditions -

- **two (2) end points**
- **slope of the curves at two (2) end points.**

and

- **one (1) intermediate point**



$$(x-a)p'_x(a,b) + (y-b)p'_y(a,b) = 0$$

Generalized CONIC

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

Re-organize:

as $\quad XSX^T = 0,\quad$ **S is symmetric**

$$\Rightarrow \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

or

$$XAX^T + GX + f = 0$$

Special Conditions:

If $b^2 = ac$, the equation represents a PARABOLA;

If $b^2 < ac$, the equation represents an ELLIPSE;

If $b^2 > ac$, the equation represents a HYPERBOLA.

SPACE CURVE (3-D)

Explicit non-parametric representation:

$$x = x, \quad y = f(x), \quad z = g(x).$$

Non-parametric implicit representation:

$$f(x, y, z) = 0, \quad g(x, y, z) = 0.$$

Intersection of the above two surfaces represents a curve.

Examples:

$$x = t^3, y = t^2, z = t.$$

A parametric space curve:

x = x(t), y = f(t), z = g(t).

Curve on the seam of a baseball:

$$x = \lambda[a.\cos(\theta + \pi/4) - b.\cos 3(\theta + \pi/4)],$$
$$y = \mu[a.\sin(\theta + \pi/4) - b.\sin 3(\theta + \pi/4)],$$
$$z = c.\sin(2\theta).$$

where,

$$\lambda = 1 + d.\sin(2\theta) = 1 + d(z/c),$$
$$\mu = 1 - d.\sin(2\theta) = 1 - d(z/c);$$
$$\theta = 2\pi t, 0 \le t \le 1.0.$$

HELIX:

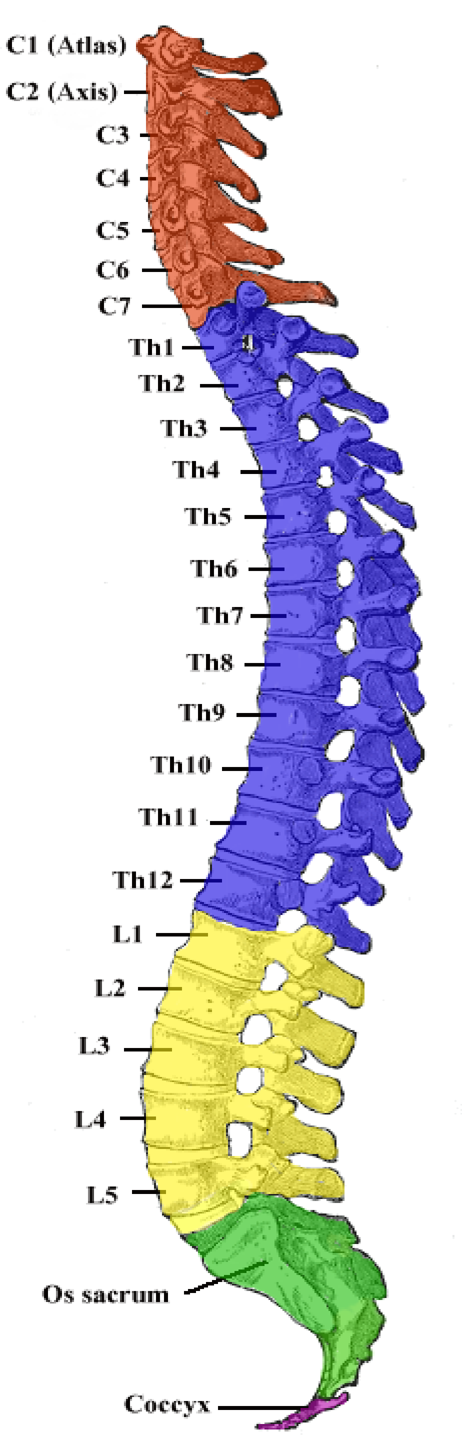
$$x = r.\cos(t), y = r.\sin(t), z = bt;$$
$$b \ne 0, -\infty < t < \infty$$

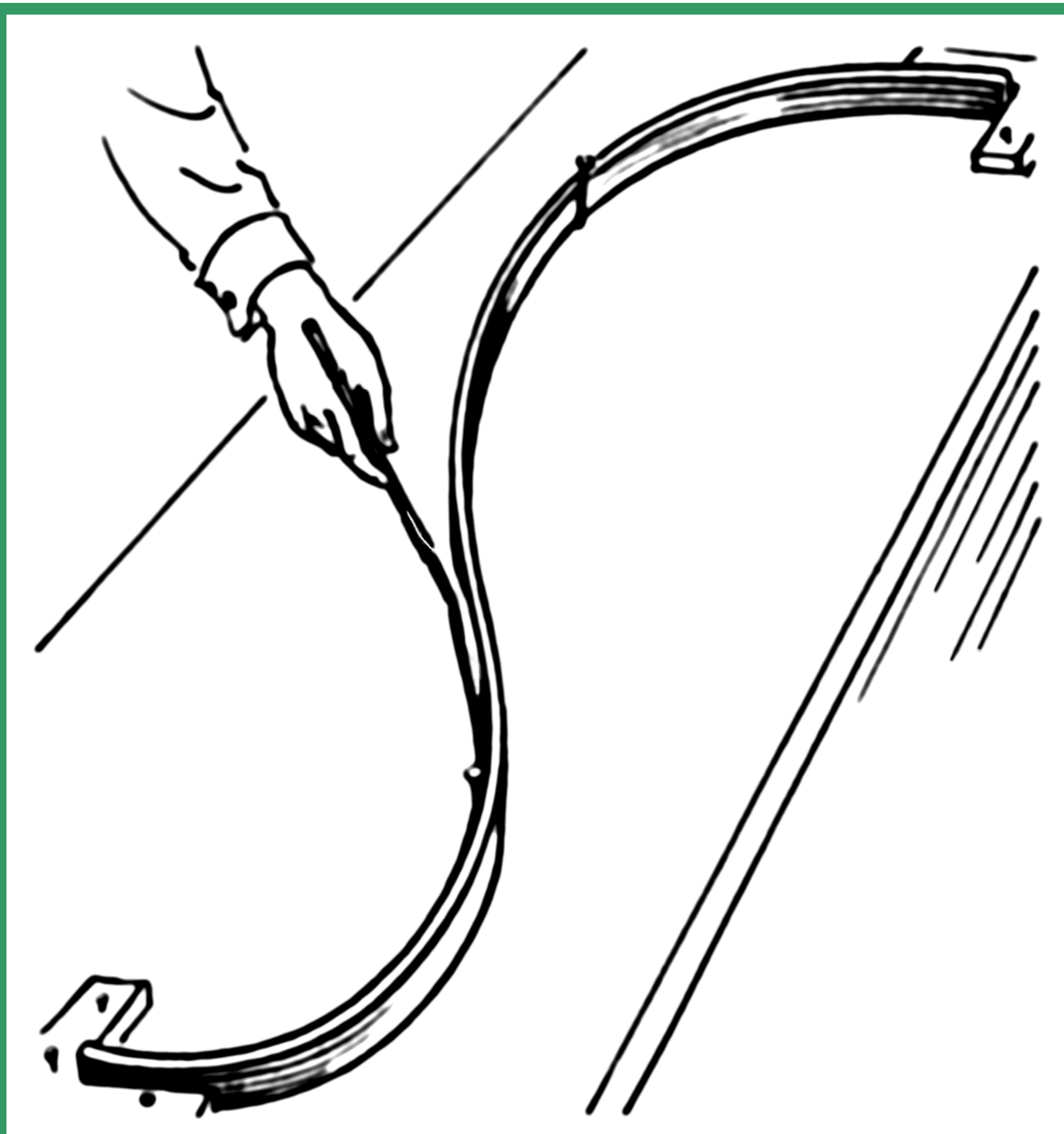
PARAMETRIC CUBIC CURVES

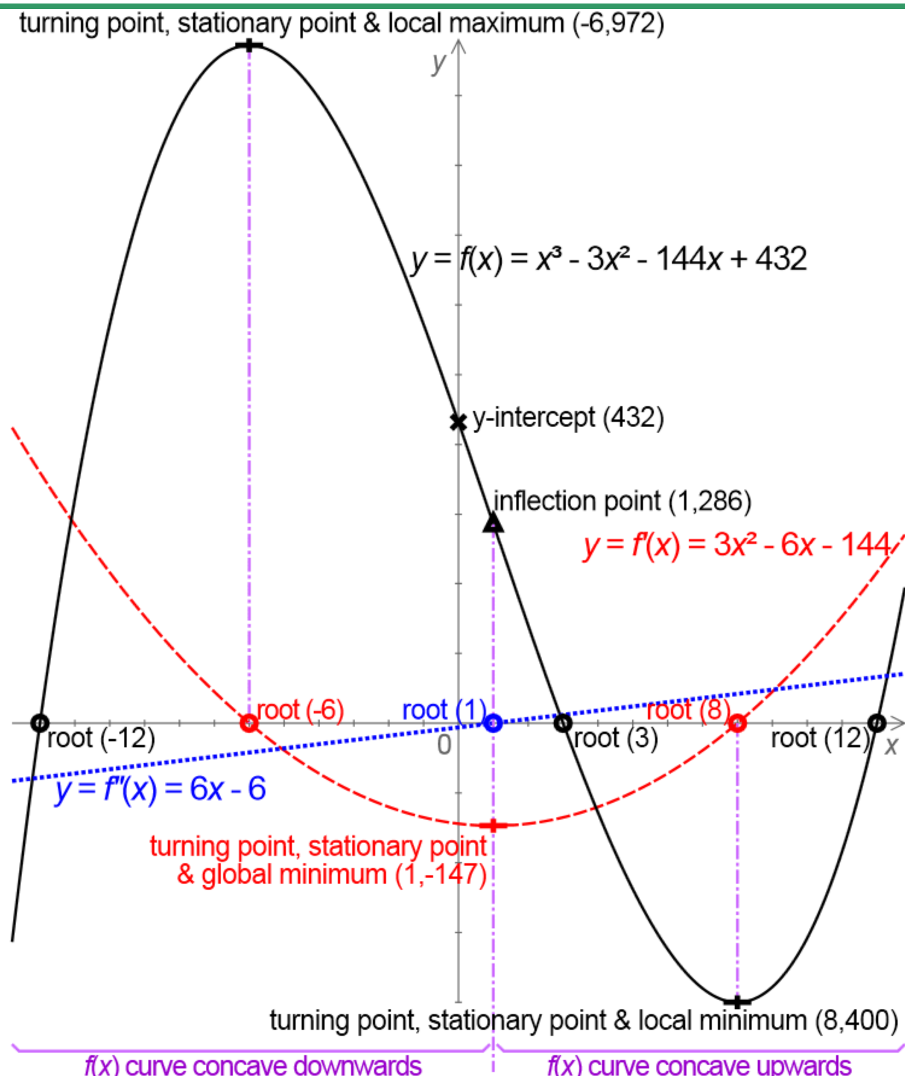
$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x,$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y,$$
$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z.$$

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T.C,$$

$$\text{where, } T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$


C1 (Atlas)
C2 (Axis)
C3
C4
C5
C6
C7
Th1
Th2
Th3
Th4
Th5
Th6
Th7
Th8
Th9
Th10
Th11
Th12
L1
L2
L3
L4
L5
Os sacrum
Coccyx





CUBIC CURVE:

- identify geonetrical/mathematical properties

PARAMETRIC CUBIC Splines

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x,$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y,$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z.$$

Spline curve refers to any composite curve, formed with Polynomial sections, satisfying specific continuity conditions (1st and 2nd derivatives) at the boundary of the pieces.

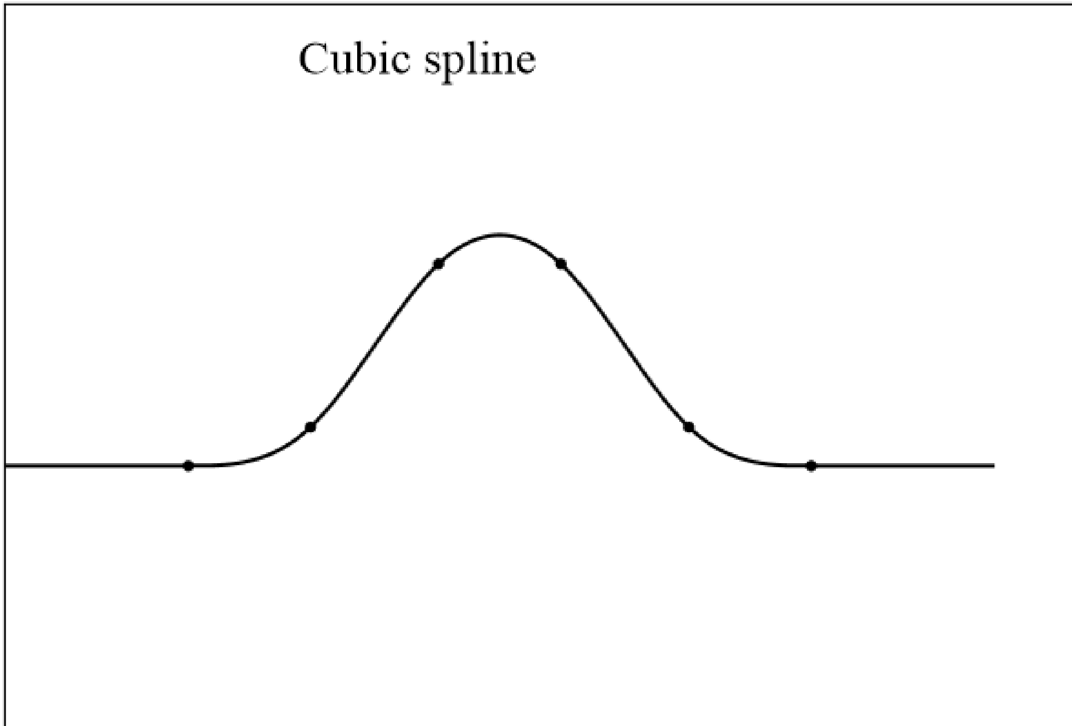
$$P(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T\,.\,CF,$$

$$where,\ T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\ and\ CF = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

To solve for:

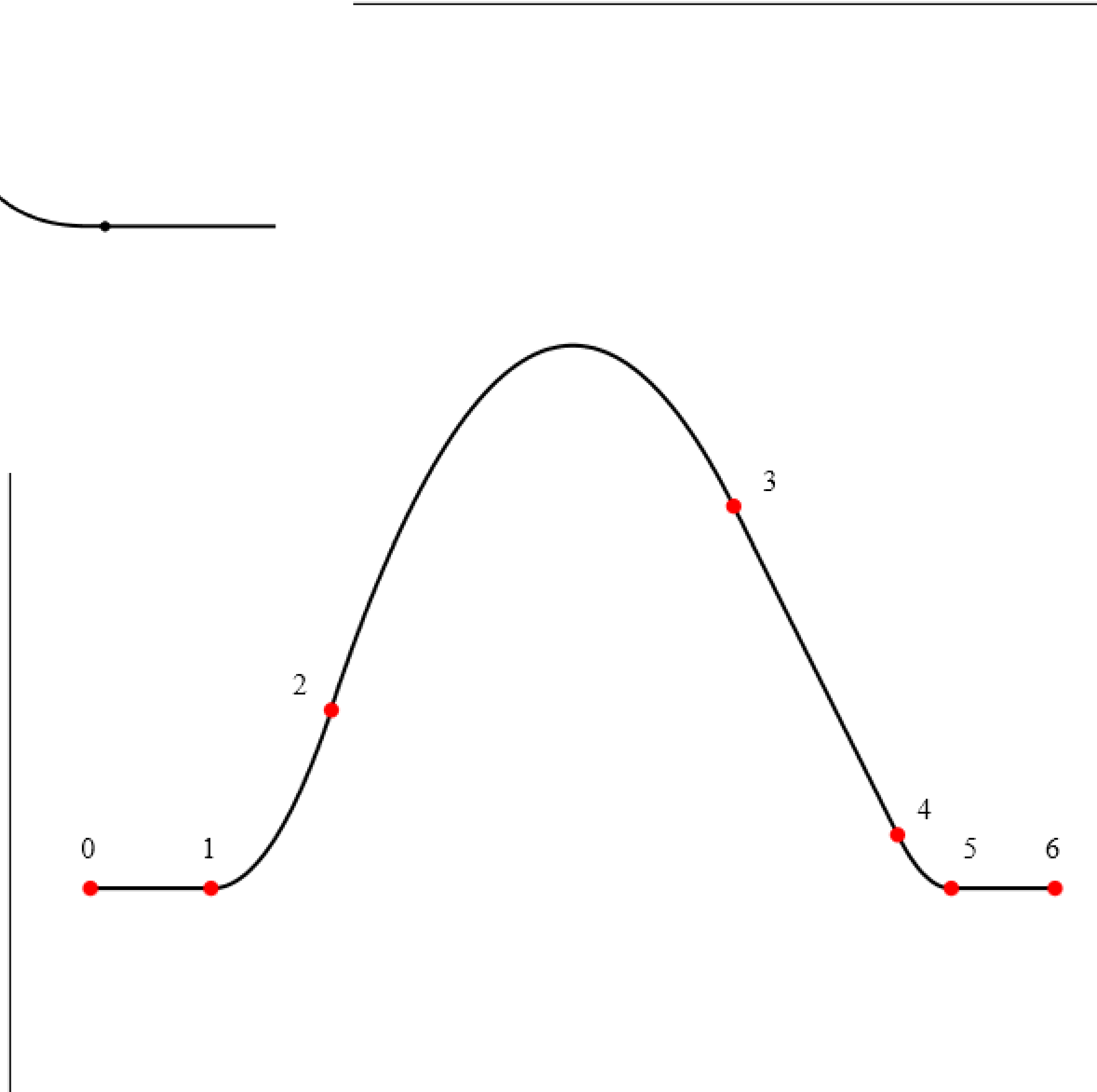
$$CF = T^{-1}P;$$

What do you need ??



With 7 Polynomial segments

Quadratic spline With 6 Polynomial segments



To solve for:

$$CF = T^{-1}P;$$

$$P(t) = [x(t) \quad y(t) \quad z(t)] = T.CF,$$

$$where,\ T = [t^3 \quad t^2 \quad t \quad 1]\ and\ CF = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

You need four (4) boundary conditions ??

$$P(t) = At^3 + Bt^2 + Ct + D;\ 0 \le t \le 1.$$

$$P(t) = [t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$P'(t) = [3t^2 \quad 2t \quad 1 \quad 0]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

Hermite Boundary Conditions:

$$P(0) = P_0; P(1) = P_1;$$

$$P'(0) = DP_0; P'(1) = DP_1;$$

$$P(t) = At^3 + Bt^2 + Ct + D;\ 0 \le t \le 1.$$

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$P'(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$P(0) = P_0; P(1) = P_1;$$

$$P'(0) = DP_0; P'(1) = DP_1;$$

Solve to get:

$$\begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = \begin{bmatrix} \boxed{} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$\begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix};$$

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = M_H G \quad (= CF);$$

In general:

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T.M.G,$$

$$\text{where, } \mathrm{T} = \begin{bmatrix} \mathrm{t}^3 & \mathrm{t}^2 & \mathrm{t} & 1 \end{bmatrix},$$

$$\mathrm{M} = \begin{bmatrix} m_{ij} \end{bmatrix}_{4x4} \text{ and } G = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \end{bmatrix}^T$$

M is a 4x4 *basis matrix* and G is a four element column vector of geometric constants, called the *geometric vector*.

The curve is a weighted sum of the elements of the geometry matrix.

The weights are each cubic polynomials of t, and are called the *blending functions*:

B = T.M.

CUBIC SPLINES

$$P(t) = \sum_{i=1}^{4} B_i t^{i-1}; t_i \le t \le t_2.$$

P(t) is the position vector of any point on the cubic spline segment.

P(t) = [x(t), y(t), z(t)]	***Cartesian***
or [r(t), θ(t), z(t)]	***Cylindrical***
or [r(t), θ(t), ϕ(t)]	***Spherical***

$$x(t) = \sum_{i=1}^{4} B_{ix} t^{i-1}$$

$$y(t) = \sum_{i=1}^{4} B_{iy} t^{i-1} \qquad t_1 \le t \le t_2.$$

$$z(t) = \sum_{i=1}^{4} B_{iz} t^{i-1}$$

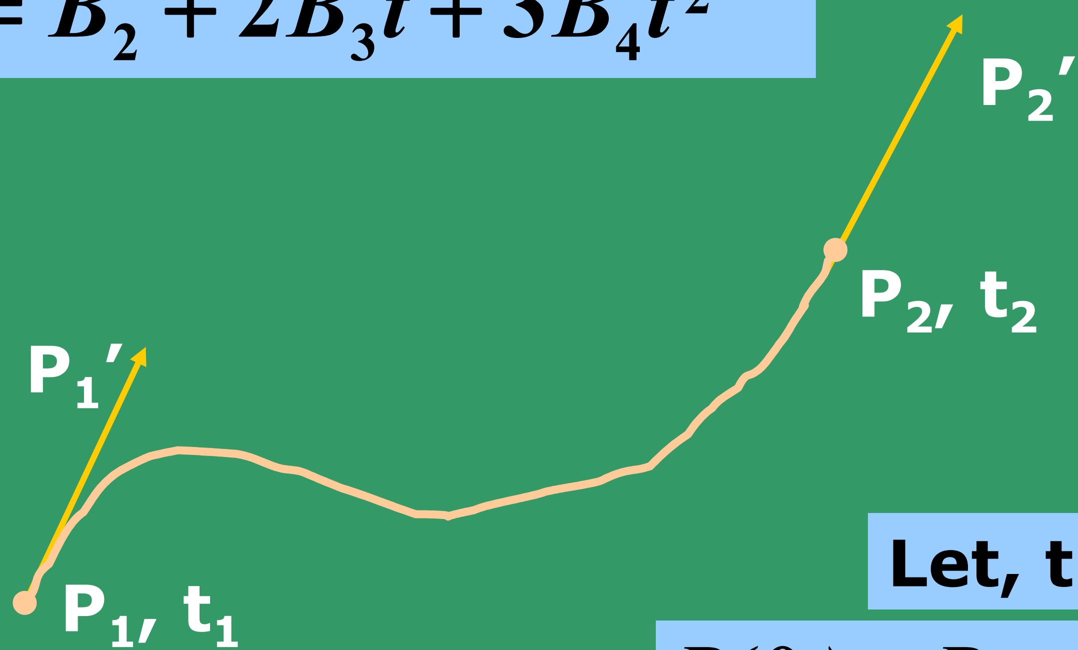
Use boundary conditions to evaluate the coeficients

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3,$$

$$t_1 \le t \le t_2$$

$$P'(t) = \sum_{i=1}^{4} (i-1) B_i t^{i-2}$$

$$= B_2 + 2B_3 t + 3B_4 t^2$$



Let, t_1=0:

$$P(0) = P_1; \quad P(t_2) = P_2.$$

$$P'(0) = P_1'; \quad P'(t_2) = P_1'.$$

Solutions:

$$B_1 = P_1;\ \ B_2 = P_1';$$

$$B_1 + B_2 t_2 + B_3 t_2^2 + B_4 t_2^3 = P(t_2);$$

$$B_2 + 2B_3 t_2 + 3B_4 t_2^2 = P'(t_2);$$

$$B_3 =$$

$$B_4 =$$

Equation of a single cubic spline segment:

$$P(t) = P_1 + P_1' t + \left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right]t^2$$

$$+ \left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right]t^3;$$

Rewrite as:

$$P(u) = \sum_{k=0}^{3} g_k H_k(u)$$

$$P(t) = P_1(2t^3 - 3t^2 + 1) + P_2(-2t^3 + 3t^2)$$

$$+ P_1'(t^3 - 2t^2 + t) + P_2'(t^3 - t^2)$$

Various other approaches used are:

- Normalized Cubic splines
- Blending
- Weighting functions.

Equation of a normalized cubic spline segment:

$$B = T.M;$$

$$P(t) = T.M.G =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \boxed{} \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}$$

Use, $t_2 = 1$;

Remember,
The derivation:

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} P(0) \\ P(1) \\ DP(0) \\ DP(1) \end{bmatrix} = M_H G \quad (= CF);$$

Equation of a single cubic spline segment:

$$P(t) = P_1 + P_1' t + \left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right] t^2 + \left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right] t^3;$$

$$P(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = T.M.G = B.G,$$

where, $\mathrm{T} = \begin{bmatrix} \mathrm{t}^3 & \mathrm{t}^2 & \mathrm{t} & 1 \end{bmatrix}$, $\mathrm{M} = \begin{bmatrix} m_{ij} \end{bmatrix}_{4x4}$

and $G = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \end{bmatrix}^T$;

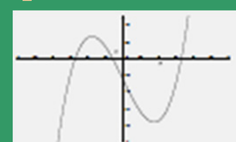
For piece-wise continuity for complex curves, two or more curve segments are joined together.

In that case, use second derivative <u>$P_2''(t)$ at end-points (joints).</u>

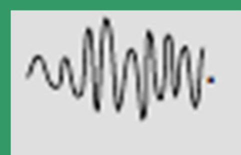
Cubic Polynomial - why and how ?

The degree three polynomial - known as a **cubic polynomial** - is the one that is most typically chosen for constructing smooth curves in computer graphics.

It is used because:
1. it is the lowest degree polynomial that can support an inflection - so we can make interesting curves, and
2. it is very well behaved numerically - that means that the curves will usually be smooth like this:

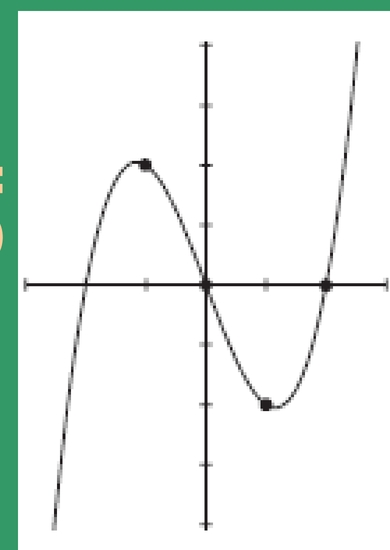


and not jumpy like this:



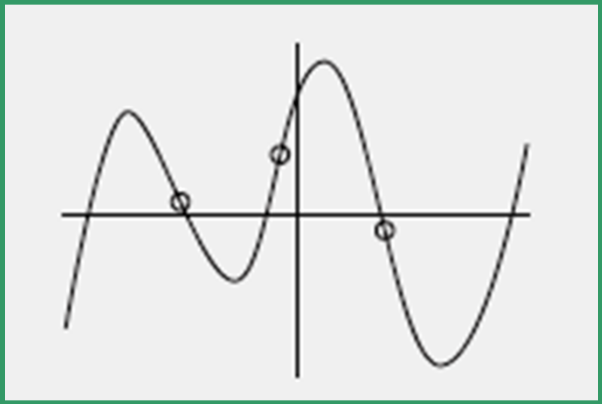
$$a + bx + cx^2 + dx^3 = y$$

control points:
(-1, 2); (0, 0); (1, -2); (2, 0)



Solution for the Coefficients can be given as:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} =$$



What do we do here – even 3rd degree is insufficient.

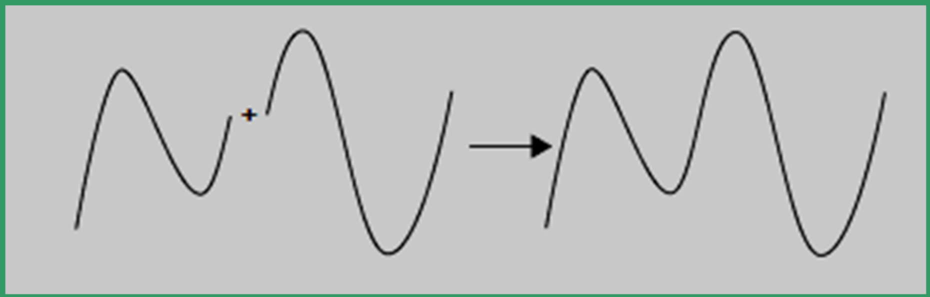
3

What about degree five, with how many extra control points ??

Three factors in the design:

- Actual Degree/order in the response of the system ??
- No. of Control Points
- Degree of the Polynomial ?

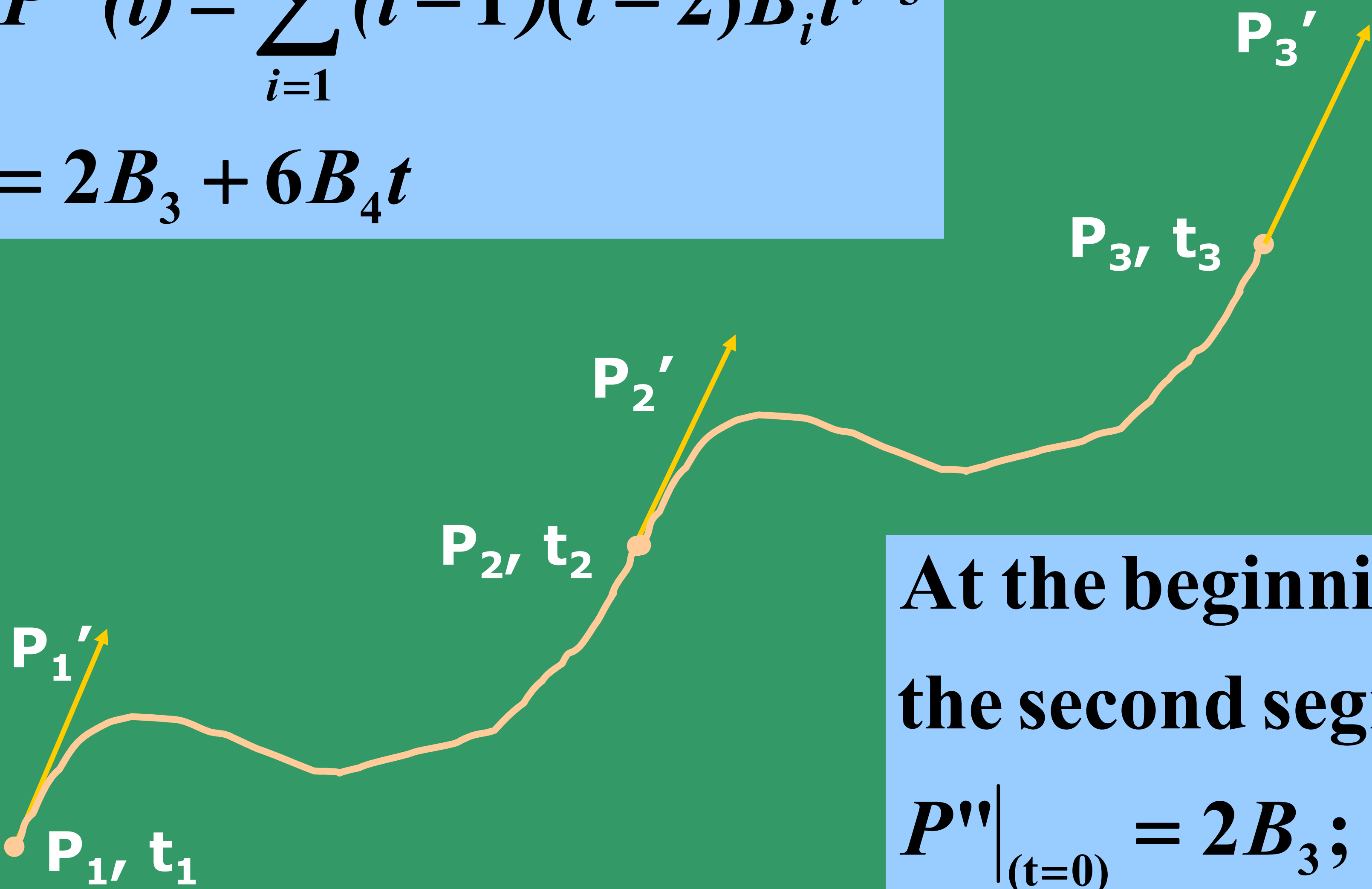
Piecewise polynomial curves:



P_1' and P_3' known,
But what about P_2' ?

$$P''(t) = \sum_{i=1}^{4} (i-1)(i-2) B_i t^{i-3}$$

$$= 2B_3 + 6B_4 t$$



At the beginning of the second segment :

$$P''\big|_{(t=0)} = 2B_3;$$

$$P''(t_2) = 2B_3 + 6B_4 t_2 = P''(0) = 2\bar{B}_3$$

$$B_3 = \frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2};$$

$$B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2};$$

$$6t_2\left[\frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}\right] + 2\left[\frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}\right] = 2\left[\frac{3(P_3 - P_2)}{t_3^2} - \frac{2P_2'}{t_3} - \frac{P_3'}{t_3}\right]$$

Multiplying both sides by $t_2 t_3$

Generalized equation for any two adjacent cubic spline segments, $P_k(t)$ and $P_{k+1}(t)$:

For first segment:

$$P_k(t) = P_k + P_k' t + \left[\frac{3(P_{k+1} - P_k)}{t_{k+1}^2} - \frac{2P_k'}{t_{k+1}} - \frac{P_{k+1}'}{t_{k+1}}\right] t^2$$

$$+ \left[\frac{2(P_k - P_{k+1})}{t_{k+1}^3} + \frac{P_k'}{t_{k+1}^2} + \frac{P_{k+1}'}{t_{k+1}^2}\right] t^3;$$

For second segment:

$$P_{k+1}(t) = P_{k+1} + P_{k+1}' t + \left[\frac{3(P_{k+2} - P_{k+1})}{t_{k+2}^2} - \frac{2P_{k+1}'}{t_{k+2}} - \frac{P_{k+2}'}{t_{k+2}}\right] t^2$$

$$+ \left[\frac{2(P_{k+1} - P_{k+2})}{t_{k+2}^3} + \frac{P_{k+1}'}{t_{k+2}^2} + \frac{P_{k+2}'}{t_{k+2}^2}\right] t^3;$$

Curvature Continuity ensured as:

$$t_{k+2} P_k' + 2(t_{k+1} + t_{k+2}) P_{k+1}' + t_{k+1} P_{k+2}' = \frac{3}{t_{k+1} t_{k+2}} \left[t_{k+1}^2 (P_{k+2} - P_{k+1}) + t_{k+2}^2 (P_{k+1} - P_k) \right]$$

Equation of a normalized cubic spline segment:

Use, $t_2 = 1$;

$$F = T.N;$$

$$P(t) = T.N.G =$$

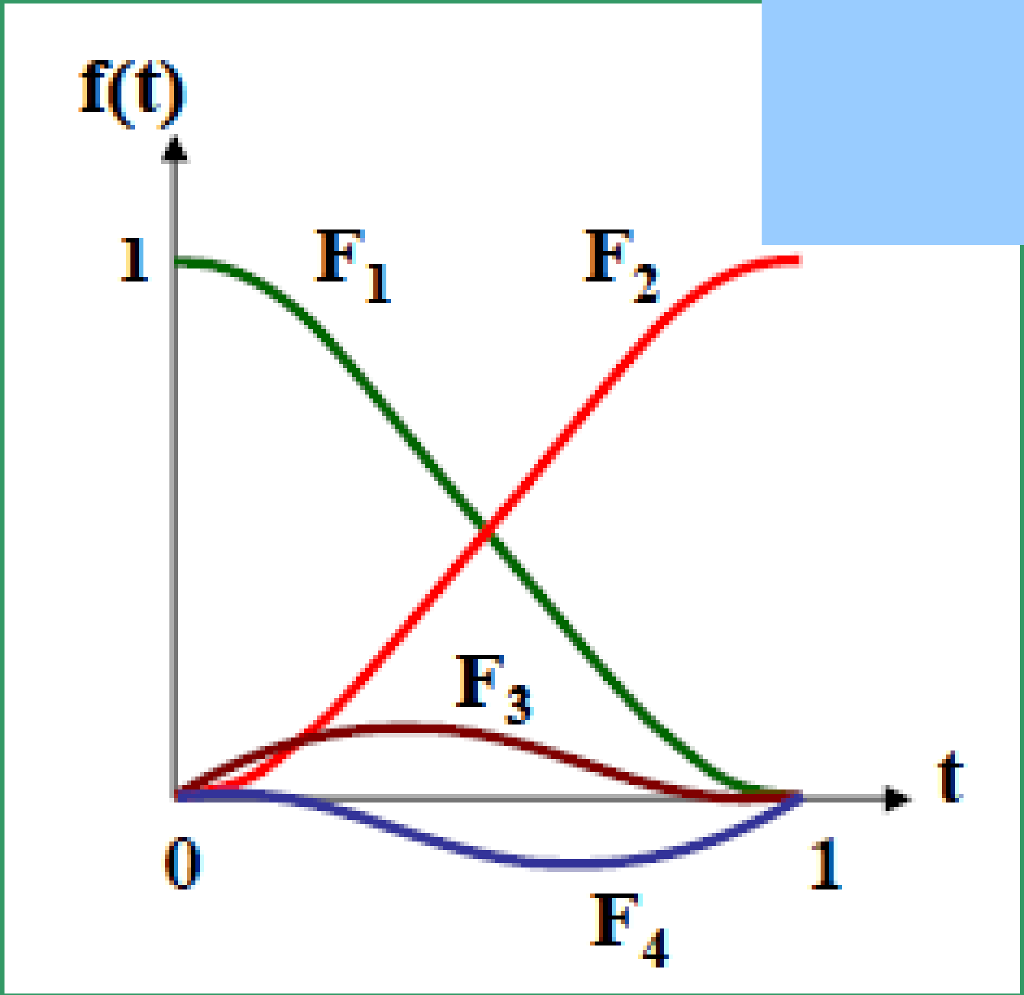
$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}$$

For curvature Continuity:

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

The Hermite Splines

$$P(t) = P_1(2t^3 - 3t^2 + 1) + P_2(-2t^3 + 3t^2) + P_1'(t^3 - 2t^2 + t) + P_2'(t^3 - t^2)$$



For curvature Continuity:

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

For three control points (knots) this works as:

$$P_2' = [3(P_3 - P_1) - P_1' - P_3']/4;$$

In general:

$$t_{k+2}P_k' + 2(t_{k+1} + t_{k+2})P_{k+1}' + t_{k+1}P_{k+2}' = \frac{3}{t_{k+1}t_{k+2}}\left[t_{k+1}^2(P_{k+2} - P_{k+1}) + t_{k+2}^2(P_{k+1} - P_k)\right]$$

For N points ??

For 3 points – 1 Eqn. (& 1 unknown)
For 4 points – 2 eqns. (& 2 unknowns)
.
.
.
For N points – (N-2) eqns. (& N-2 unknowns)

Write the eqn. set for N = 5; in matrix form.

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & \cdot & \\ & & \cdot & \cdot & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_n \end{bmatrix}.$$

$$= \begin{bmatrix} P_1' \\ \frac{3}{t_2 t_3}\left[t_2^2(P_3 - P_2) + t_3^2(P_2 - P_1)\right] \\ \frac{3}{t_3 t_4}\left[t_3^2(P_4 - P_3) + t_4^2(P_3 - P_2)\right] \\ \cdot \\ \cdot \\ \cdot \\ \frac{3}{t_{n-1} t_n}\left[t_{n-1}^2(P_n - P_{n-1}) + t_n^2(P_{n-1} - P_{n-2})\right] \\ P_n' \end{bmatrix}$$

Solve using: Forward-backward substitution:

Thomas Algm.

$$P_k' + 4P_{k+1}' + P_{k+2}' = 3[P_{k+2} - P_k]$$

Lets solve for N = 4;

$$P_1' + 4P_2' + P_3' = 3[P_3 - P_1];$$

$$P_2' + 4P_3' + P_4' = 3[P_4 - P_2]$$

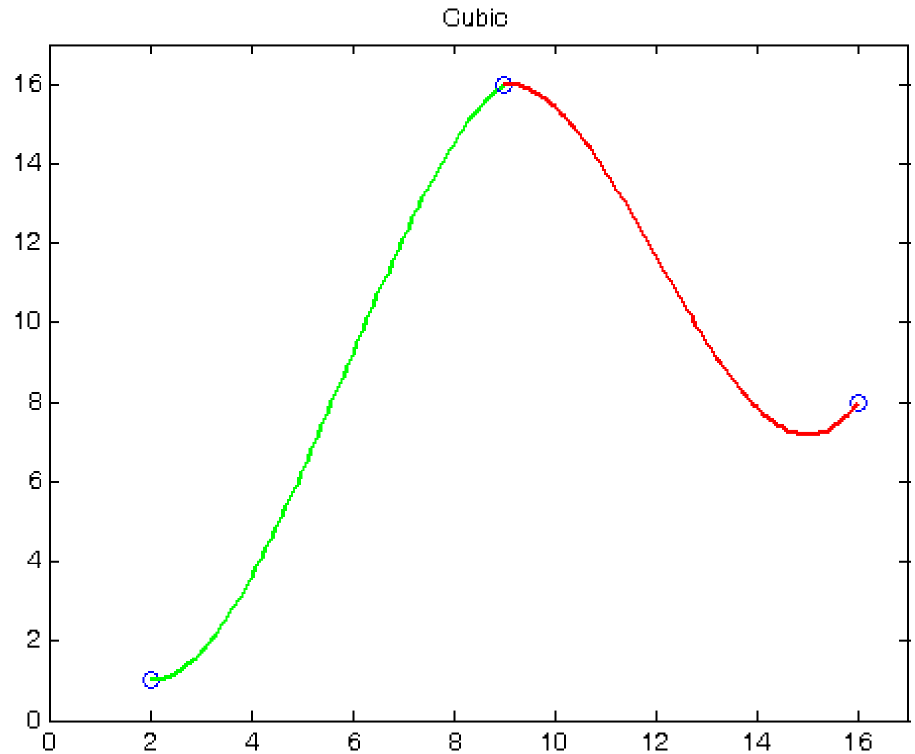
Re-arrange to get:

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} P_2' \\ P_3' \end{bmatrix} = \begin{bmatrix} 3(P_3 - P_1) - P_1' \\ 3(P_4 - P_2) - P_4' \end{bmatrix};$$

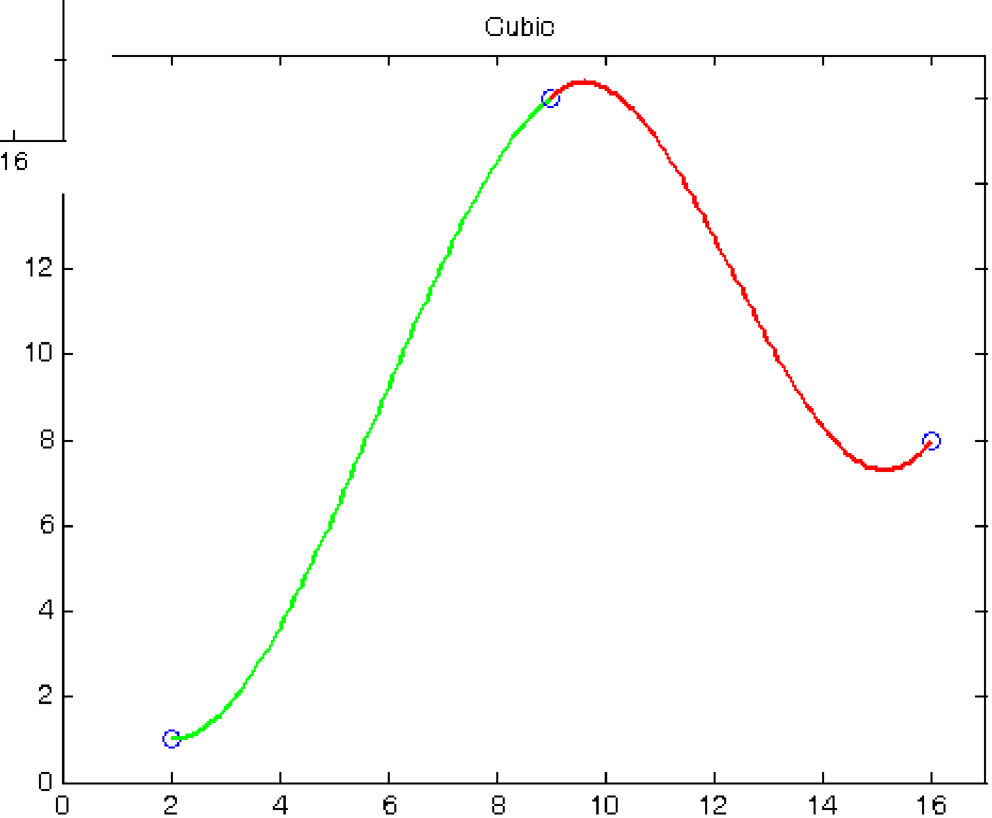
$$\begin{bmatrix} P_2' \\ P_3' \end{bmatrix} = (1/15) \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 3(P_3 - P_1) - P_1' \\ 3(P_4 - P_2) - P_4' \end{bmatrix}$$

Problem: **The position vectors of a normalized cubic spline are given as (0 0), (1 1), (2 -1) and (3 0). The tangent vectors at the ends are both (1 1).**

Soln: **The 2 internal tangent vectors are calculated, and both are equal to (1 –0.8).**

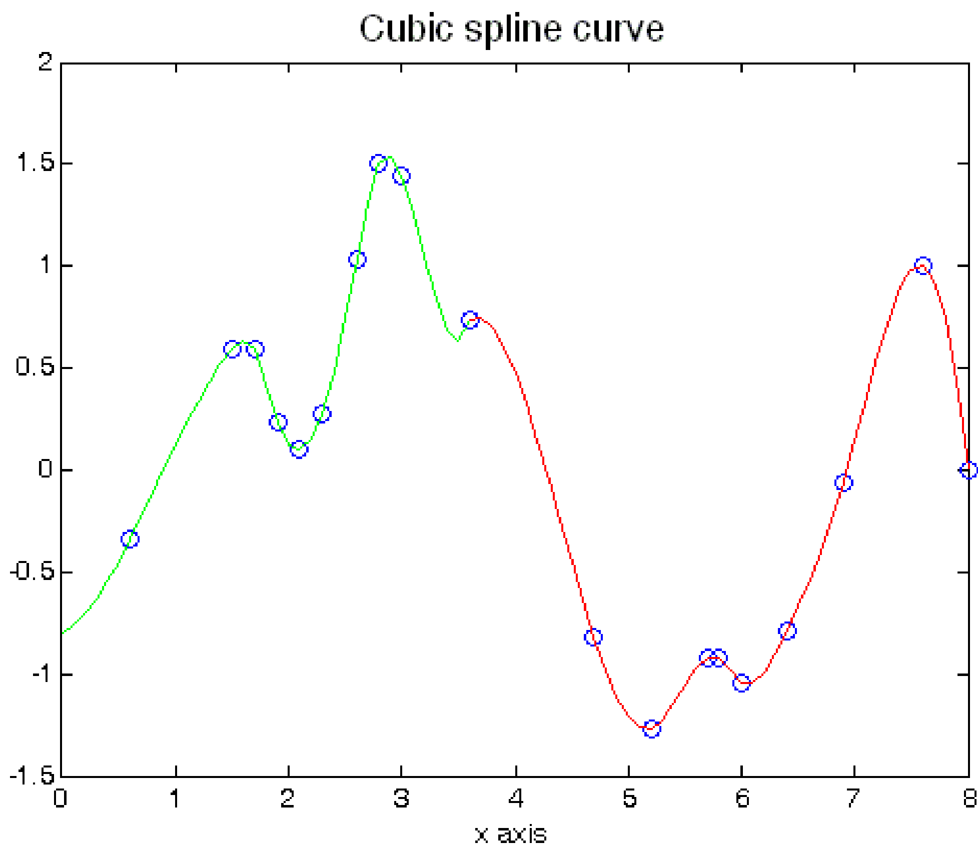


No use of 2nd derivative smoothing



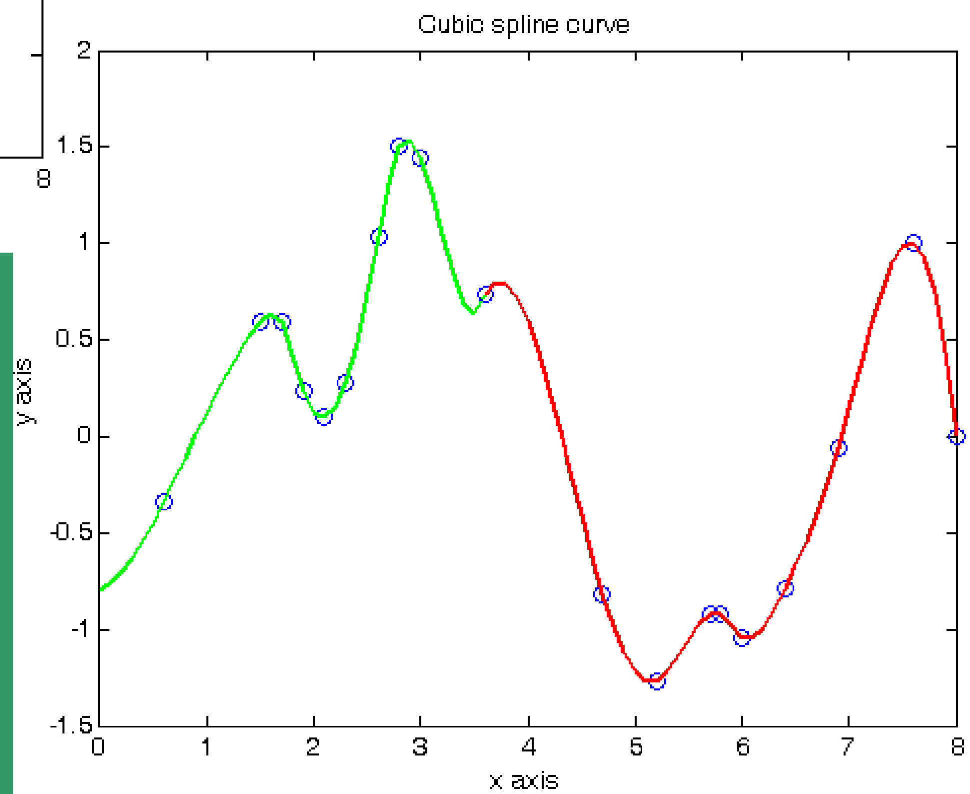
Two piecewise cubic spline segments

Using 2nd derivative smoothing



No use of 2^{nd} derivative smoothing

Examples of spline interpolation



Using 2^{nd} derivative smoothing

Other Variants:

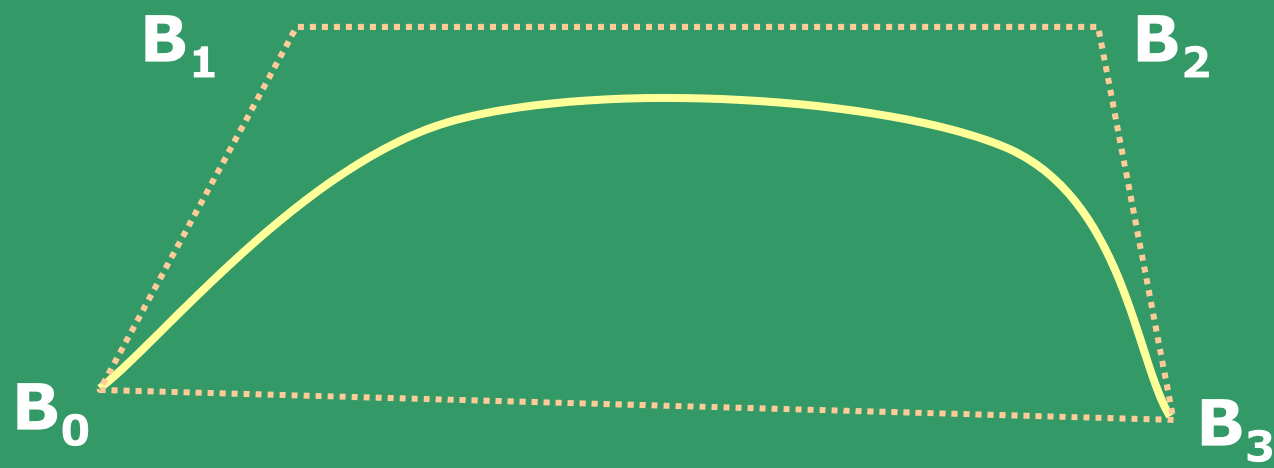
- **Cardinal Splines;**
- **Catmul-Rom splines**
- **Irvine-Hall Splines**
- **T-spline**
- **B-spline**

BEZIER CURVES

- Basis functions are real
- Degree of polynomial is one less than the number of points
- Curve generally follows the shape of the defining polygon
- First and last points on the curve are coincident with the first and last points of the polygon
- Tangent vectors at the ends of the curve have the same directions as the respective spans
- The curve is contained within the convex hull of the defining polygon
- Curve is invariant under any affine transformation.

A few typical examples of cubic polynomials for Bezier

BEZIER CURVES



Equation of a parametric Bezier curve:

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t); \; 0 \le t \le 1$$

B_i's are called the *control points*;

where the **Bezier or Bernstein basis** or **blending function** is:

Binomial Coefficients:

(*i*th, *n*th-order **Bernstein basis** function)

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$J_{n,i}(t)$ is the ith, nth order Bernstein basis function.

n is the degree of the defining Bernstein basis function (polynomial curve segment).

This is one less than the number of points used in defining Bezier polygons.

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t); \quad 0 \le t \le 1$$

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Limits for $i = 0:$

0^0=1; 0! = 1

$$J_{n,0}(0) = \frac{n!}{0!n!} 0^0 (1-0)^{n-0} = 1;$$

For $i \ne 0:$ $J_{n,i}(0) = \dfrac{n!}{i!(n-i)!} 0^i (1-0)^{n-i} = 0;$

Also:

$$J_{n,n}(1) = 1, i = n;$$

$$J_{n,i}(1) = 0, i \neq n.$$

Thus:

$$P(0) = B_0 J_{n,0}(0) = B_0.$$

$$P(1) = B_n J_{n,n}(1) = B_n.$$

For any t:

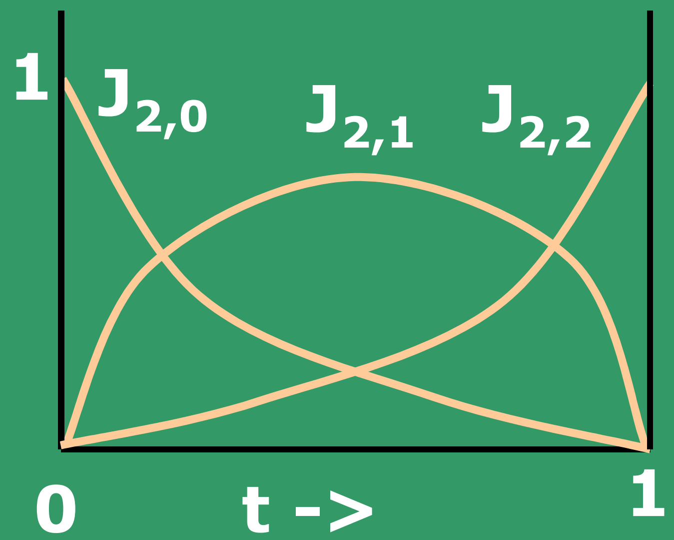
$$\sum_{i=0}^{n} J_{n,i}(t) = 1$$

Also Verify:

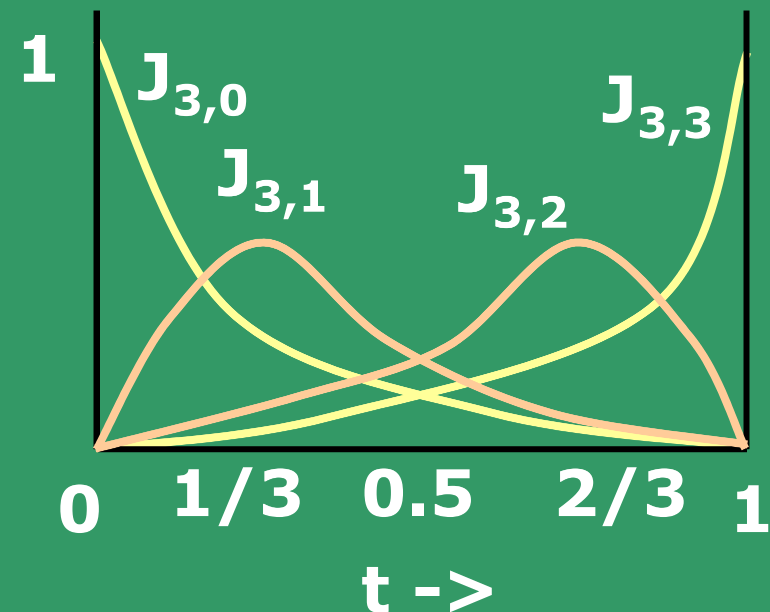
$$J_{n,i}(t) =$$

$$(1-t).J_{(n-1),i}(t) + t.J_{(n-1),(i-1)}(t); \;\; n > i \geq 1$$

Below are some examples of BBF (Bezier /Bernstein blending functions:



n = 2



n = 3 (cubic)

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}; \qquad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Take n = 3:

$$\binom{n}{i} = \binom{3}{i} = \frac{6}{i!(3-i)!}$$

$$J_{3,0}(t) = 1.t^0(1-t)^3 = (1-t)^3;$$

$$J_{3,1}(t) = 3.t.(1-t)^2;$$

$$J_{3,2}(t) = 3.t^2.(1-t);$$

$$J_{3,3}(t) = t^3.$$

Thus, for Cubic Bezier:

$$\boldsymbol{P(t)} = \boldsymbol{(}1-\boldsymbol{t)}^3\boldsymbol{B}_0 + 3\boldsymbol{t(}1-\boldsymbol{t)}^2\boldsymbol{B}_1 + 3\boldsymbol{t}^2\boldsymbol{(}1-\boldsymbol{t)B}_2 + \boldsymbol{t}^3\boldsymbol{B}_3$$

$$= \begin{bmatrix} \boldsymbol{t}^3 & \boldsymbol{t}^2 & \boldsymbol{t} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_0 \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_3 \end{bmatrix} ;\boldsymbol{n} = 3.$$

For Cubic-splines:

$$P(t) = T.N.G =$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}^T$$

For n = 4:

$$P(t) = \begin{bmatrix} t^4 & t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

$$= T.N.G = F.G;$$

where:

$$F = [J_{n,o}(t) \;\; J_{n,1}(t) \;\; \ldots\ldots\; J_{n,n}(t)]$$

$$N = \left\lfloor \lambda_{ij} \right\rfloor_{nxn}$$

where:

$$\lambda_{ij} = \begin{cases} \dbinom{\boldsymbol{n}}{\boldsymbol{j}}\dbinom{\boldsymbol{n-j}}{\boldsymbol{n-i-j}}(-1)^{n-i-j} & 0 \le (\mathrm{i}+\mathrm{j}) \le \mathrm{n} \\ 0 & \text{otherwise} \end{cases}$$

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

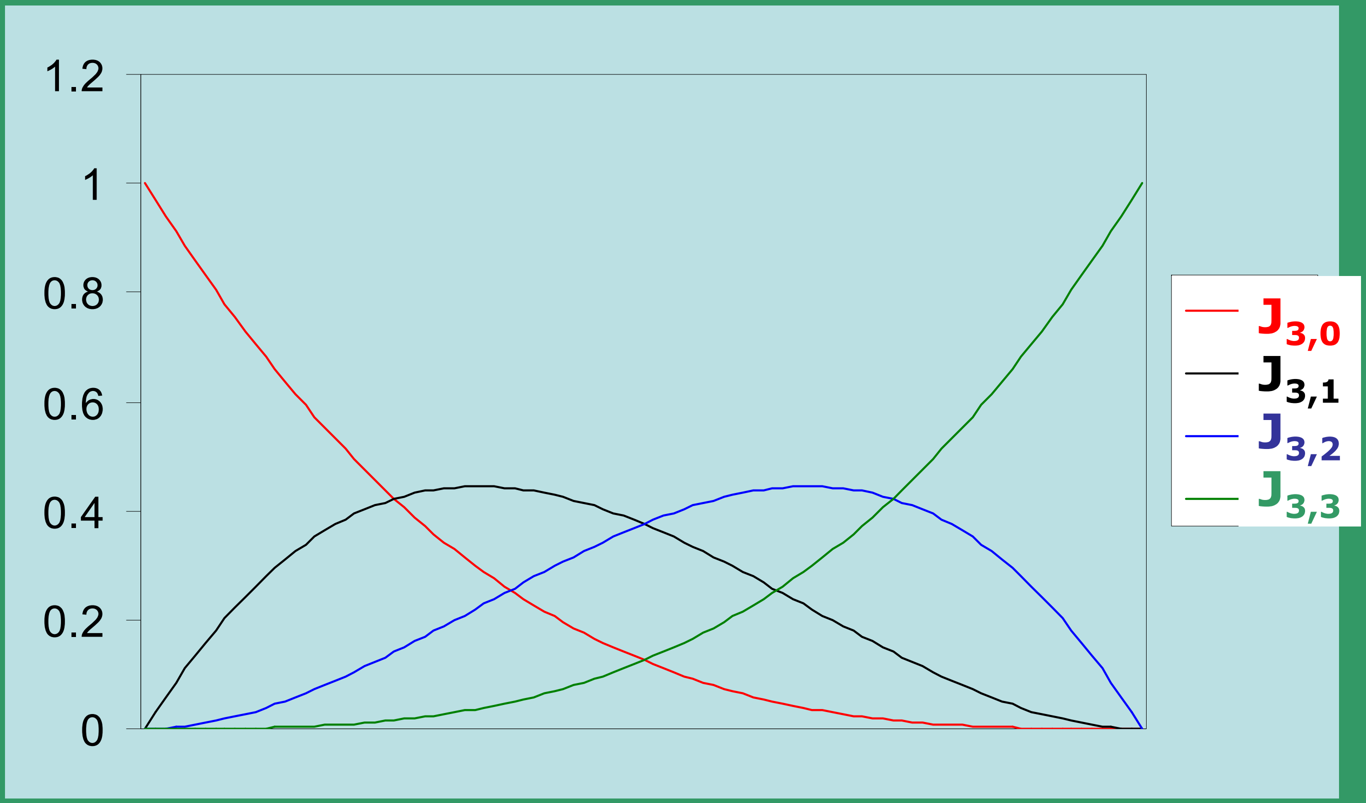
Computation of successive binomial coefficients:

$$\binom{n}{i} = (\text{[redacted]}) \binom{n}{i-1}$$

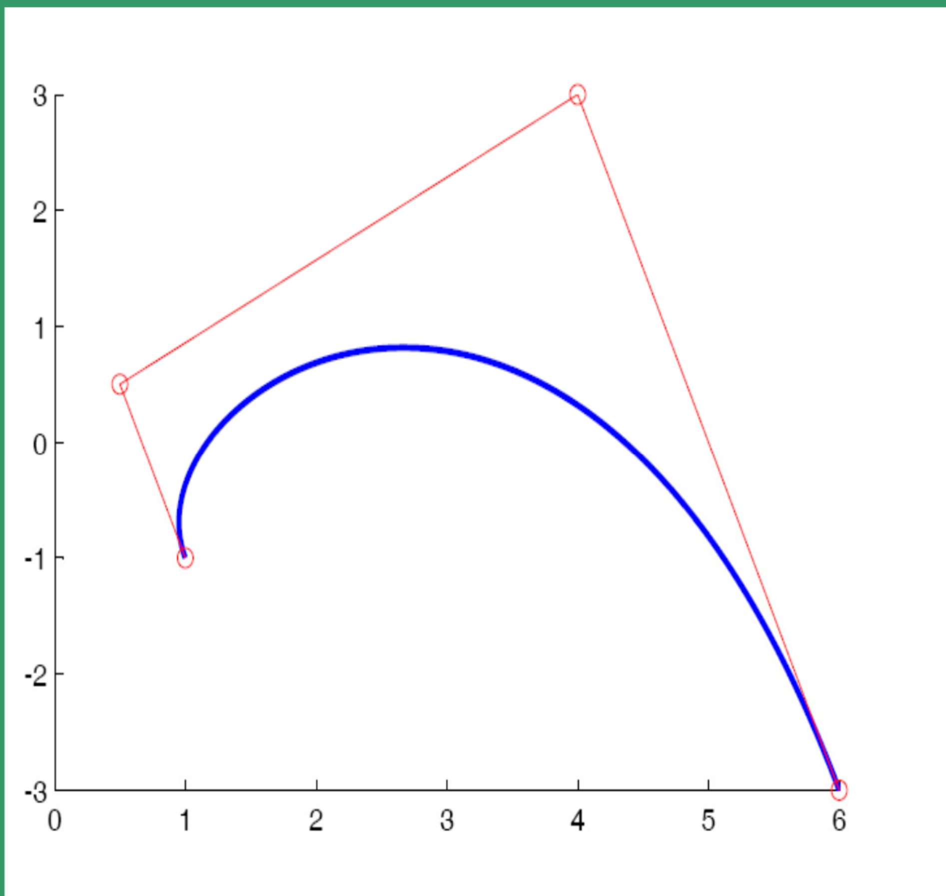
Home Assignment:

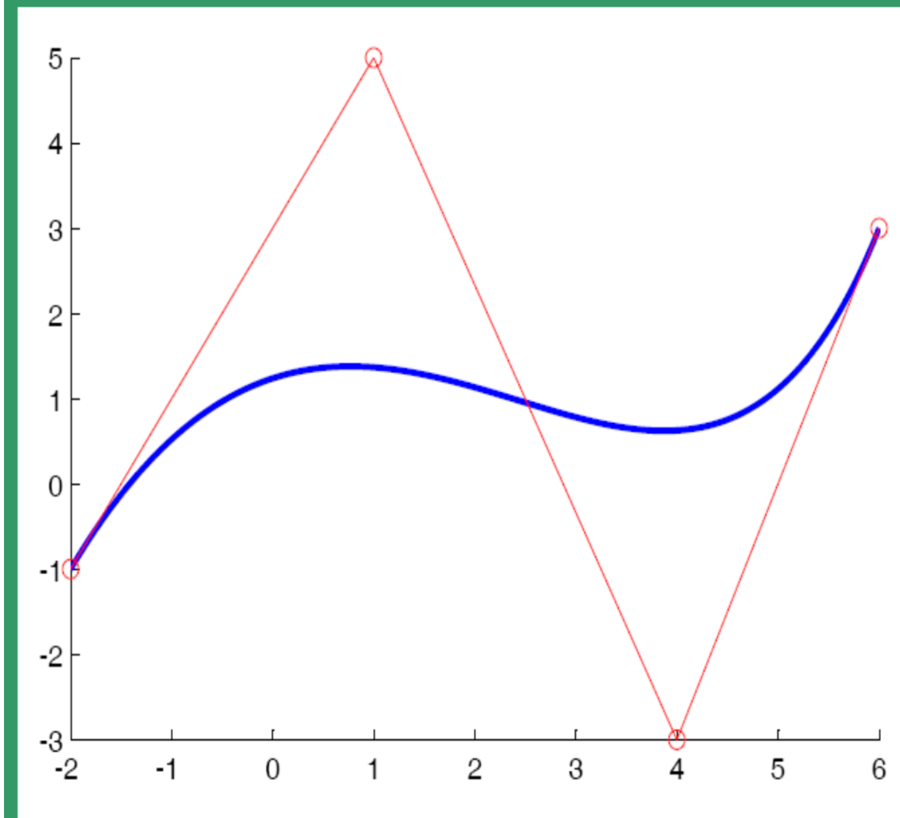
Get the expressions of $J_{2,i}$ and $J_{4,i}$

Bezier Basis Functions

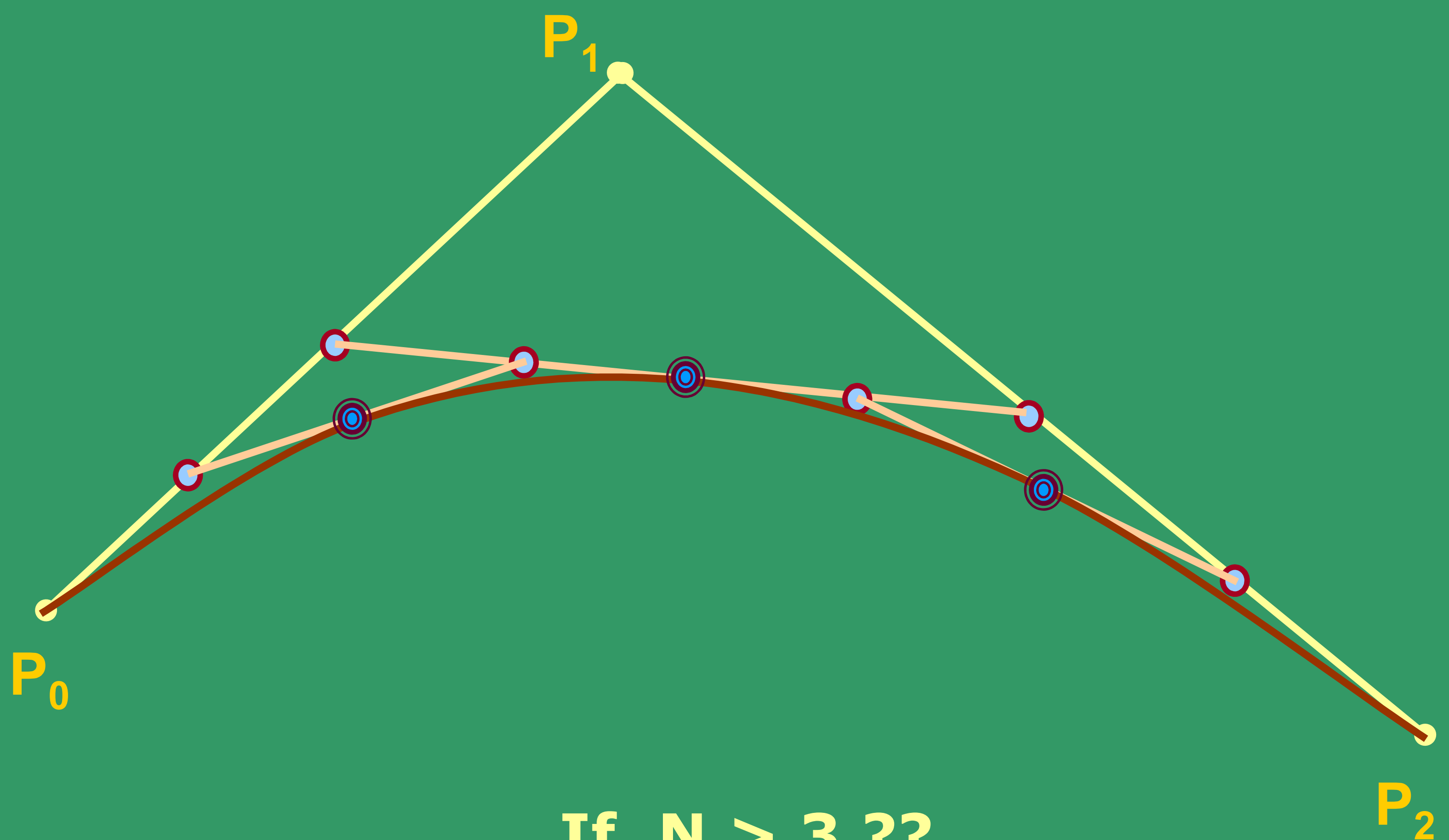


Bezier Curve Examples



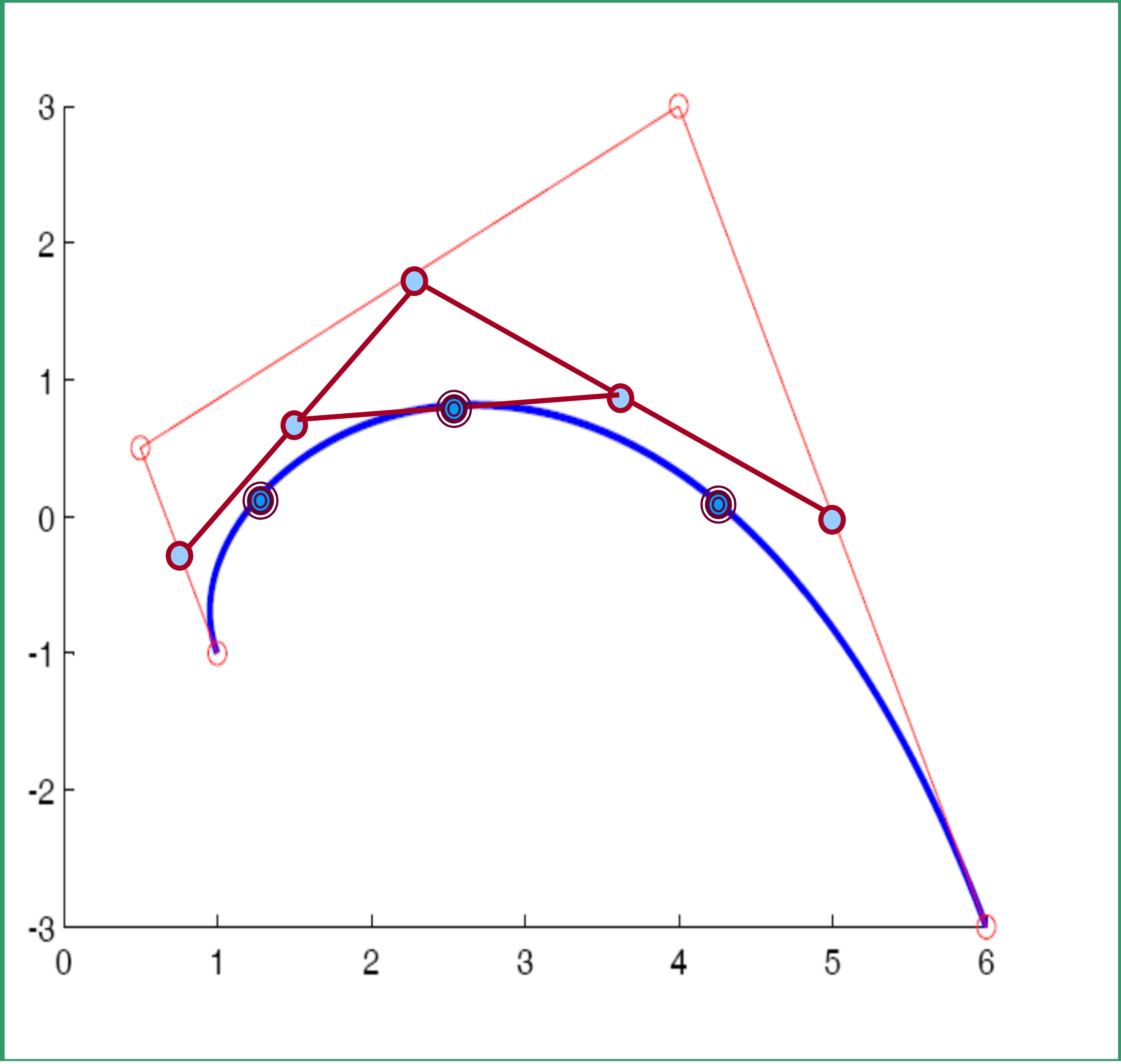


Recursive geometric definition of BEZIER CURVES

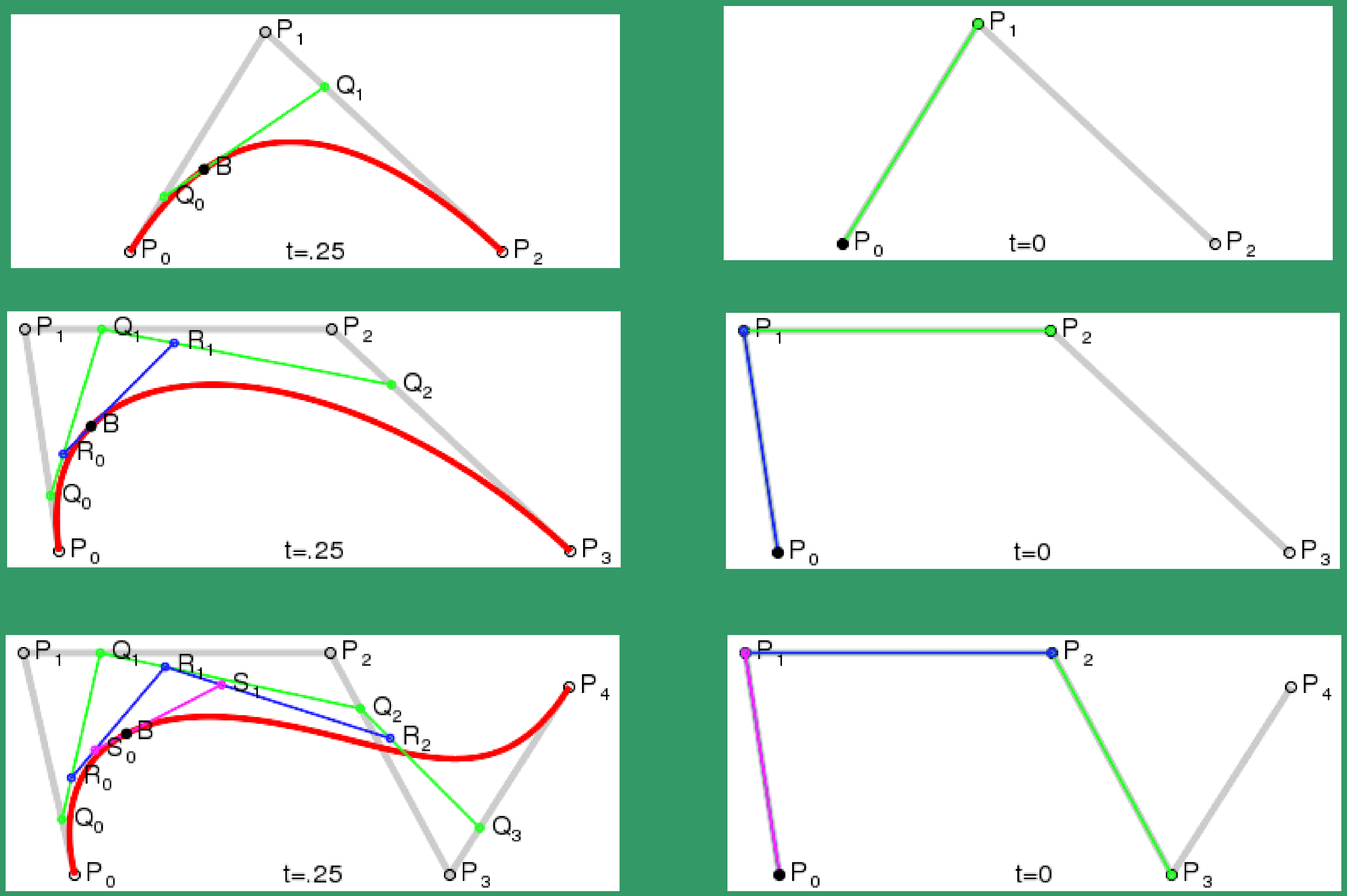


If, N > 3 ??

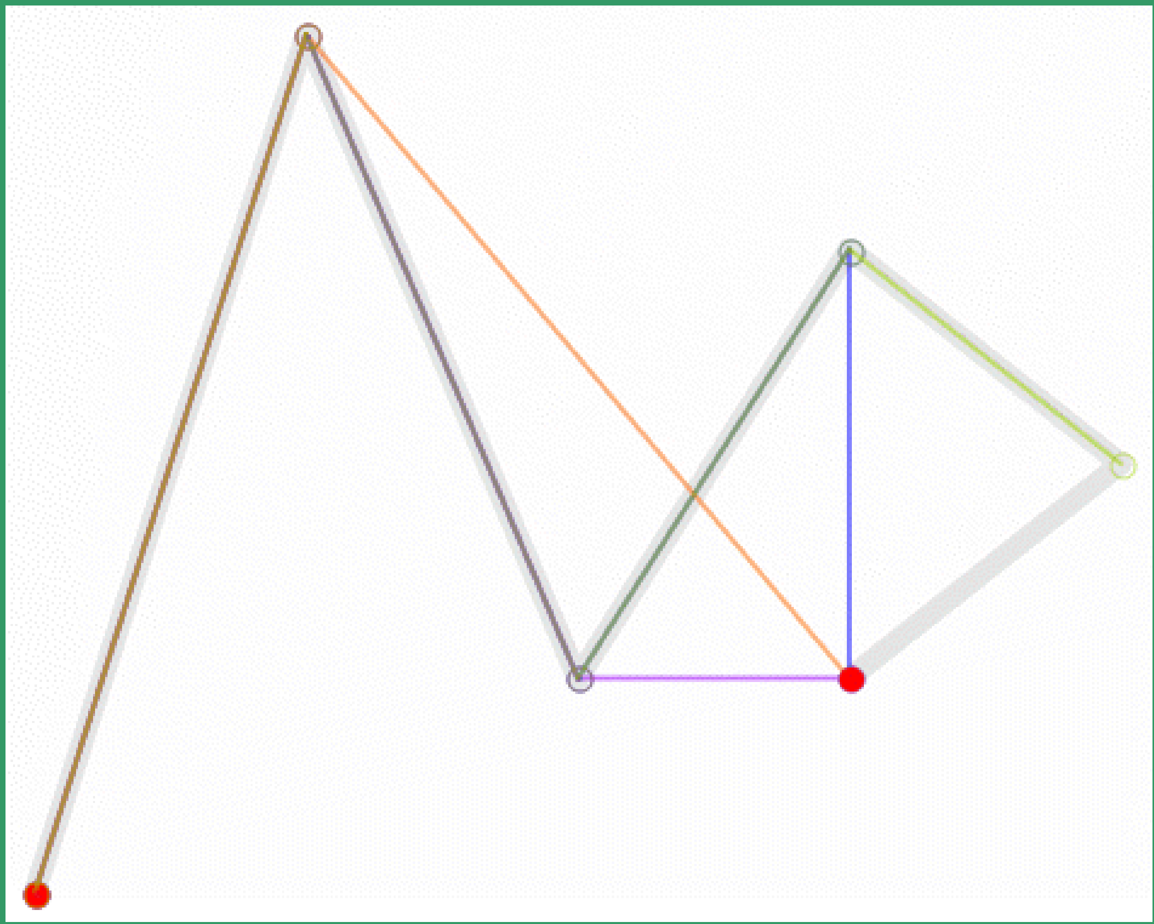
Recursive Bezier Curve Example



Iterative Bezier Curve Animation



Iterative Higher-order Bezier Curve Animation



More to follow:

- **B-splines represented as blending functions**
- **Conversion between one format to another.**
- **Knots and control points.**
- **When B-spline becomes a Bezier?**

QUADRICS – 3-D analogue of conics:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Jz + K = 0$$

Basis Splines (B-splines):

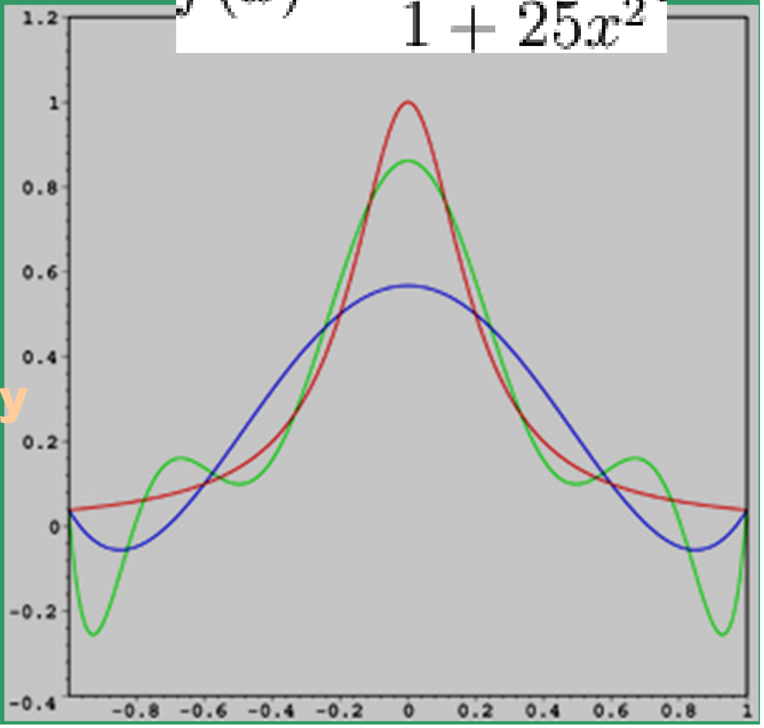
- a generalisation of a Bézier curve, avoids the Runge phenomenon without increasing the degree of the B-spline

The red curve is the Runge (The Cauchy–Lorentz distribution or Breit–Wigner distribution) function.

The blue curve is a 5th-order interpolating polynomial (using six equally-spaced interpolating points).

The green curve is a 9th-order interpolating polynomial (using ten equally-spaced interpolating points).

$$f(x) = \frac{1}{1+25x^2}.$$



At the interpolating points, the error between the function and the interpolating polynomial is (by definition) zero.

Between the interpolating points (especially in the region close to the endpoints 1 and −1), the error between the function and the interpolating polynomial gets worse for higher-order polynomials.

Limitations of Bezier Curves:

- Not enough flexibility

- Higher degree with more No. of control points

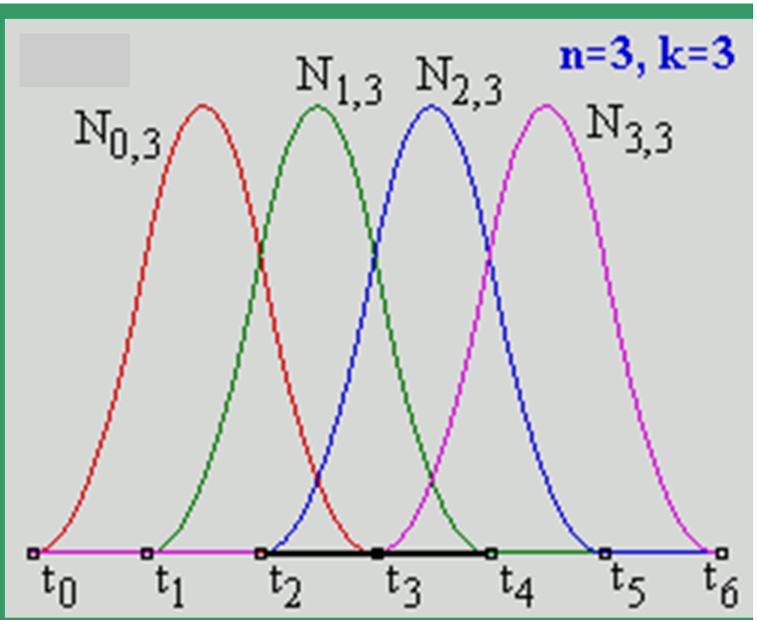
- Larger degree has instability, numerical errors, and computational costly

- Not enough local control (global effect of change).

-

In mathematics, a spline is a special function defined piece-wise by polynomials.

Spline interpolation is often preferred to polynomial interpolation because it yields similar results, even when using low-degree polynomials, while avoiding Runge's phenomenon for higher degrees.



Periodic uniform B-spline basis, with k= 3, p = 3; Uniform Knots: [0 1 2 3 4 5 6];

$N_{i,k}$ (i-th B-spline blending function, of order k) is a polynomial of order *k* (degree *k-1*) on each interval:

$$t_i < t < t_{i+k}.$$

k must be at least 2 (linear) and can be not more, than *p+1* (the number of control points = n in Fig. above).

A <u>knot vector</u> $(t_0, t_1, \ldots, t_{p+k})$ must be specified. Across the knots basis, functions are C^{k-2} continuous.

The form of a B-spline curve is very similar to that of a Bézier curve. However, unlike a Bézier curve, a B-spline curve involves more information, namely:

- **a set of *p* <u>control points</u>,**
- **a <u>knot vector</u> of *m* knots, and**
- **a degree *n* *(i.e. order n+1)*.**

Note that *n*, *m* and *p* must satisfy $m = n + p + 1$. More precisely, if we want to define a B-spline curve of degree *n* with *p* control points, we have to supply $n + p + 1$ knots:

$$t_0, t_1, \ldots, t_{n+p+1}.$$

On the other hand, if a knot vector of *m* knots and p control points are given, the degree of the B-spline curve is:

$n = m - p - 1$ or $m - (p+1)$.

Basis Splines (B-splines):

- **Degree is independent of the No. of control Points**
- **Local Control over Shape**
- **More complex than Bezier**

Given *m* values $t_i \in [0,1]$, called *knots*, with $t_0 \le t_1 \le \cdots \le t_{m-1}$

a B-spline of degree *n* is a parametric curve $\mathbf{S} : [t_n, t_{m-n-1}] \to \mathbb{R}^d$

composed of linear combination of basis B-splines $b_{i,n}$
(of degree n):

$$\mathbf{S}(t) = \sum_{i=0}^{m-n-2} \mathbf{P}_i b_{i,n}(t) \ , t \in [t_n, t_{m-n-1}]$$

$1 \le n \le p;$

/* unnecessary

The P_i are called control points or de Boor points (there are *m-n-1* control points). A polygon can be constructed by connecting the de Boor points with lines, starting with P_0 and finishing with P_{m-n-2}. This polygon is called the de Boor polygon.

The *m-n-1* basis B-splines of degree *n* for n = 0,1,…,(m-2), can be defined using the Cox-de Boor recursion formula:

$$b_{j,0}(t) := \begin{cases} 1 & \text{if} \quad t_j \le t < t_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

j = 0,1,…,(m-2)

$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t).$$

$$t \in [t_{j,} t_{j+n+1}]$$

j = 0,1,…,(m-n-2)

(j+n+1) can not exceed m-1, which limits both j and n.

The above recursion formula specifies how to construct nth-order function from two B-spline function of order (n-1).

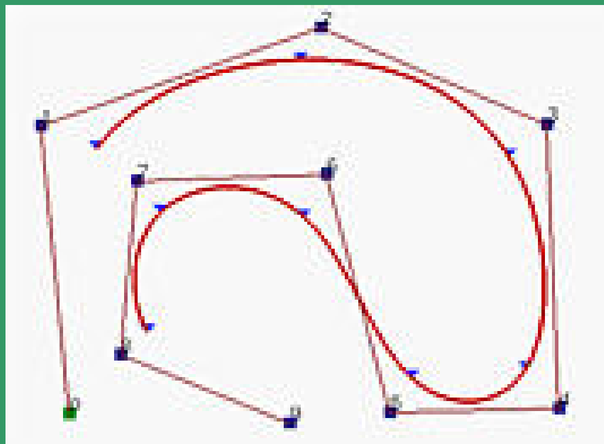
No. of Control Points: (m – n - 1);

Degree of Spline: n;

(m-n-1=4=n+1; n=3) - If B-spline has [0 0 0 0 1 1 1 1] knot vector, we get Bezier basis.

No. of Knots: m (= No. of Control Points + degree + 1);

B-splines



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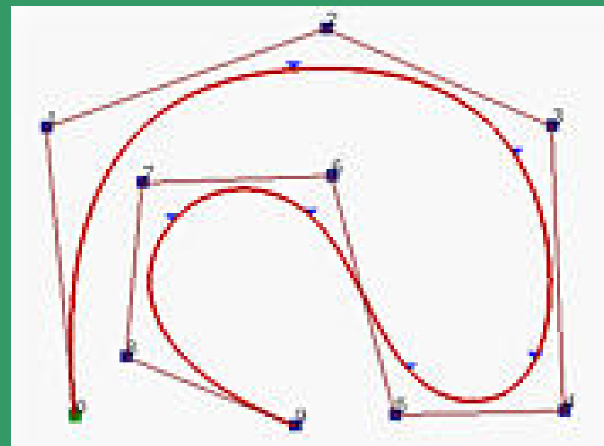
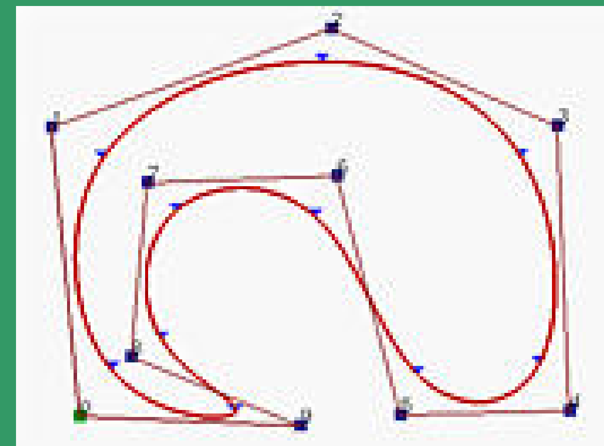


CLAMPED



CLOSED

The above figures have p control points (p=10) and n = 3. Then, m must be 14, so that the knot vector has 14 knots.

To have the clamped effect, the first n+1 = 4 and the last 4 knots must be identical. The remaining 14 - (4 + 4) = 6 knots can be anywhere in the domain (giving non-periodic structure).

In fact, the central curve is generated with knot vector: U = { 0, 0, 0, 0, 0.14, 0.28, 0.42, 0.57, 0.71, 0.85, 1, 1, 1, 1 }.

Note that except for the first four and last four knots, the middle ones are almost uniformly spaced. In fact, the little triangles are the knot points. Periodic structure gives closed curves. Avoid multiplicty at ends for open unclamped curves.

The "**Standard Knot Vector**" for a B-spline of order (n + 1) begins and end with a knot of "multiplicity" (n+1) and uses unit spacing for the remaining knots.

Let, No. of control points: m-n-1 = 8;
and for a cubic (n=3) B-spline: n + 1 = 4;

So, m = 12; The "**Standard Knot Vector**" is"

[0 0 0 0 1 2 3 4 5 5 5 5]

Periodic, Cubic B-spline Blending functions :

$$B_{0,3}(t) = (1-t)^3 / 6;$$

$$B_{1,3}(t) = (3.t^3 - 6t^2 + 4) / 6;$$

$$B_{2,3}(t) = (-3.t^3 + 3t^2 + 3t + 1) / 6;$$

$$B_{3,3}(t) = t^3 / 6.$$

$B_{i,n}(t)$ is non-zero only in the interval: $t \in [t_i, t_{i+n+1}]$

Hence it spans the knots:

$$t_{i,} t_{i+1,} t_{i+2,......,} t_{i+n+1}]$$

The recursion for integer knots

$$(n)B_{jn}(t) =$$
$$(t-j)B_{j,n-1}(t) + (n+1+j-t)B_{j+1,n-1}(t)$$

$$b_{j,n}(t) := \frac{t-t_j}{t_{j+n}-t_j} b_{j,n-1}(t) + \frac{t_{j+n+1}-t}{t_{j+n+1}-t_{j+1}} b_{j+1,n-1}(t).$$

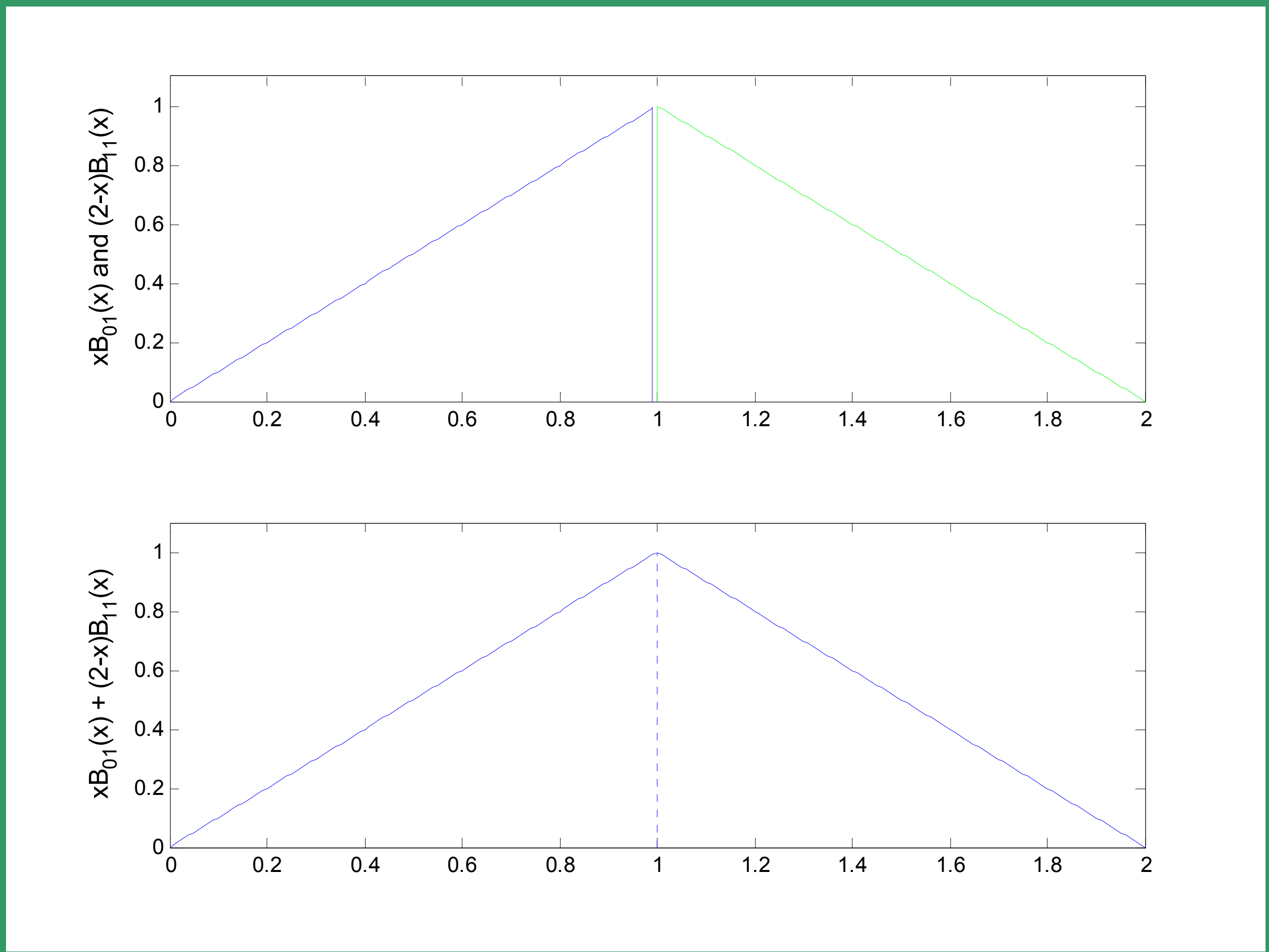
Lets solve for, the B-spline function of order 2 (degree n=1) beginning at n=0, the recursion is ??

$$B_{01}(t) =$$

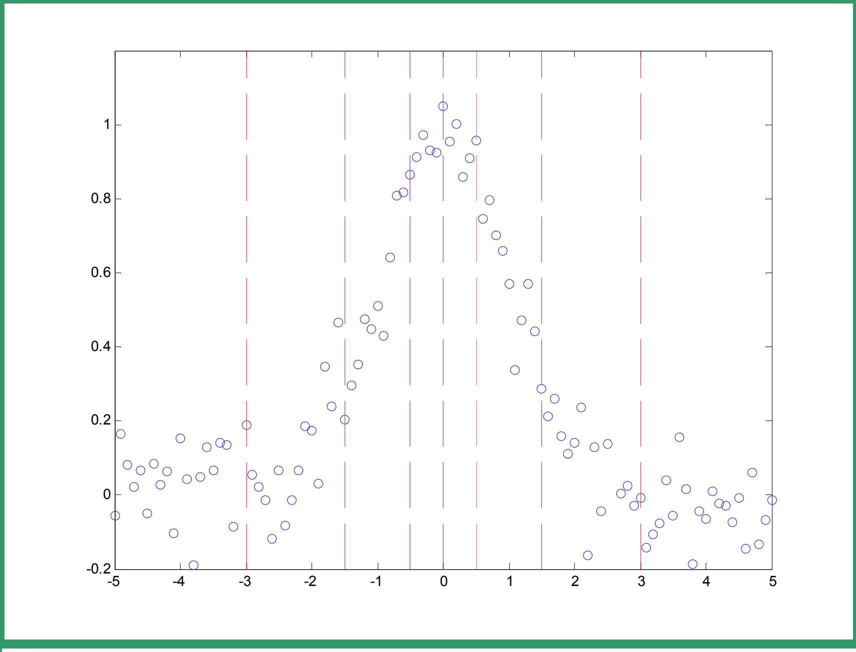
Degree is "n" and order is "m" = n + 1.

$$b_{j,0}(t) = 1_{[t_j,t_{j+1}]} = \begin{cases} 1 & \text{if} \quad t_j \le t < t_{j-1} \\ 0 & \text{otherwise} \end{cases}$$

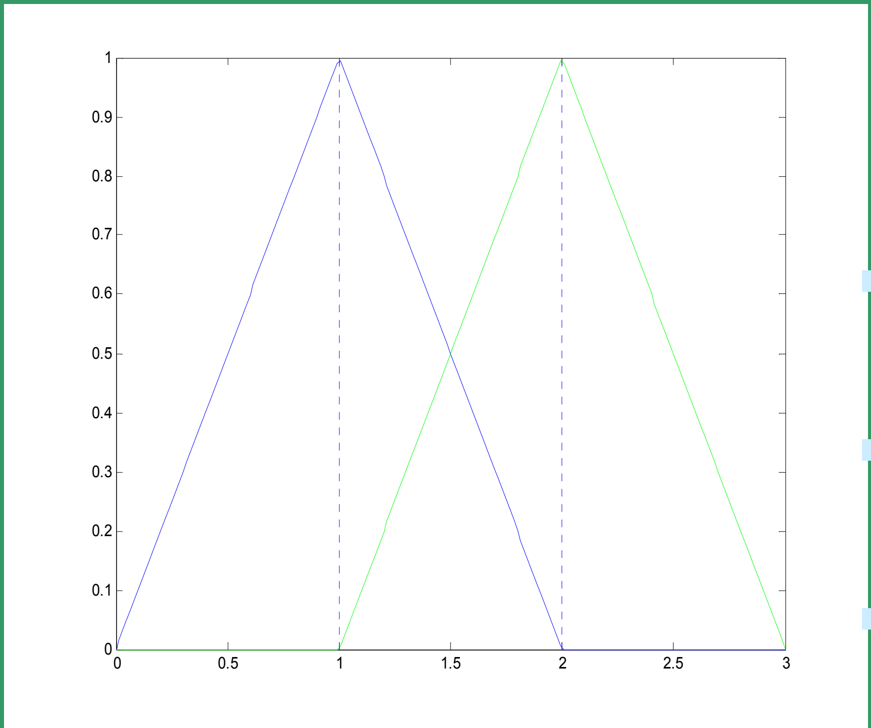
Now Plot $B_{01}(t)$ from Two Boxes $B_{00}(t)$ and $B_{10}(t)$



$B_{01}(t)$ is a tent function



knots: $\xi_0, \xi_1, \ldots, \xi_L$.



- B-splines of order 2 are tent functions, starting at a knot, rising linearly to 1 at the next knot, and decaying linearly to 0 two knots over.
- They ($B_{0,1}$ & $B_{1,1}$) are continuous.
- Order 2 implies a continuous derivative of order 0.
- Order 2 knots are piecewise linear

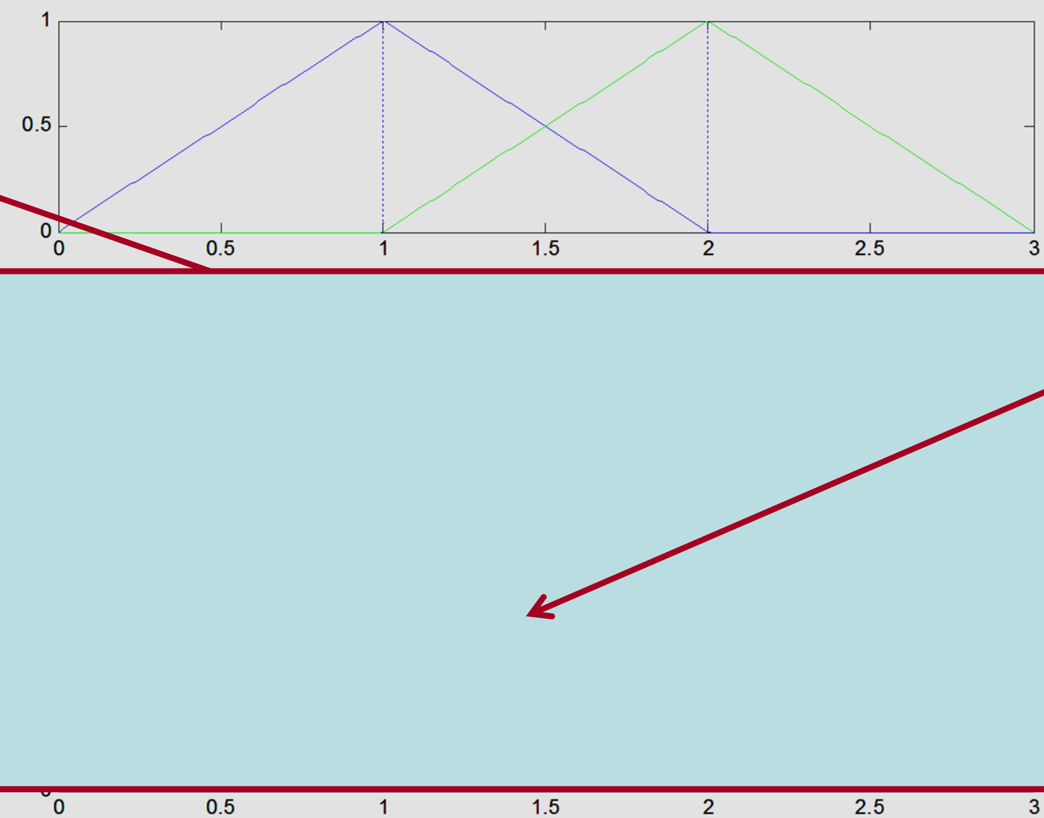
Order 3 - $B_{02}(t)$ from Two Tent Functions

$$B_{0,2}(t) =$$

$t.(2-t)$

$(3-t)(t-1$

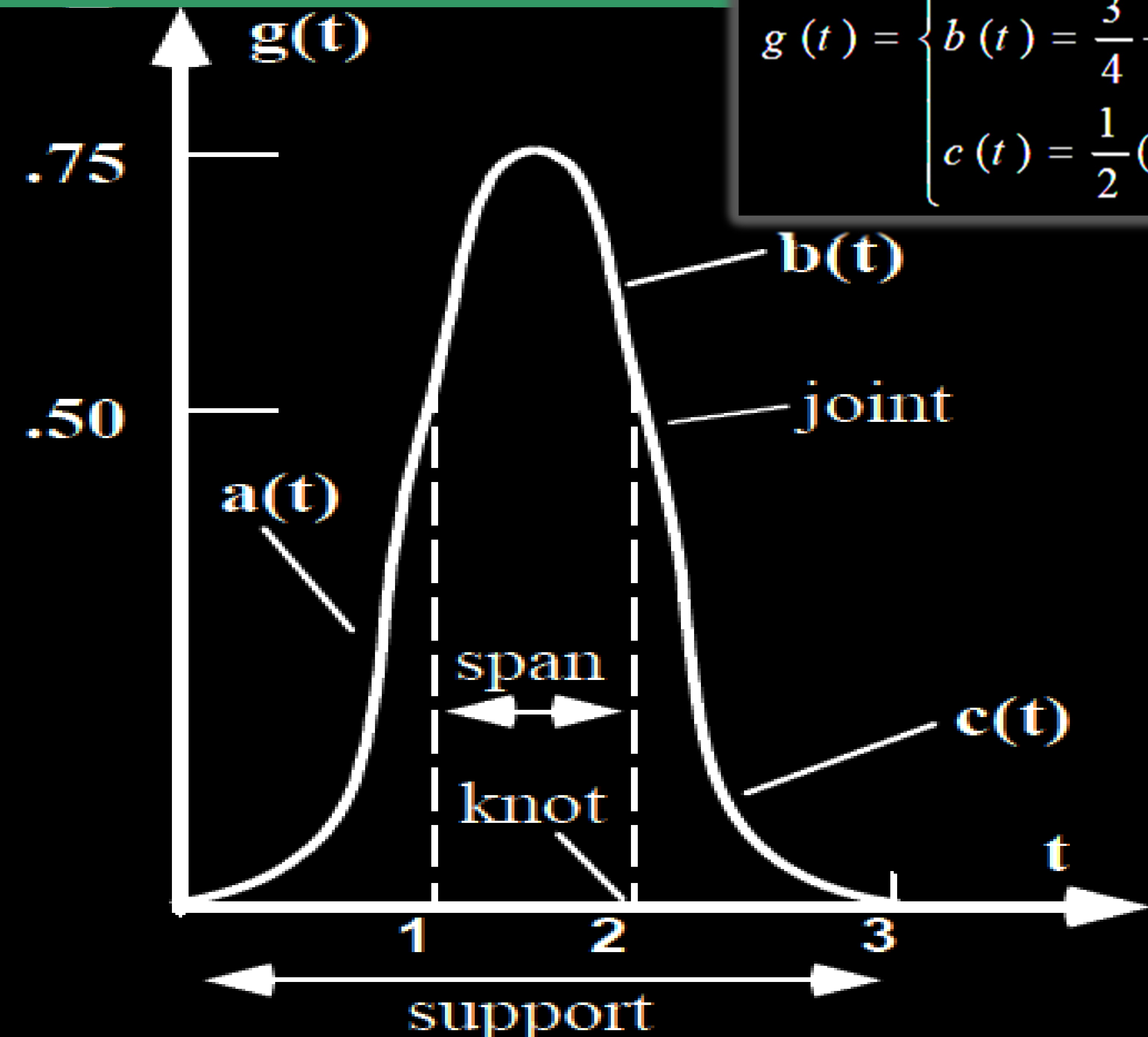
$(3-t)^2$

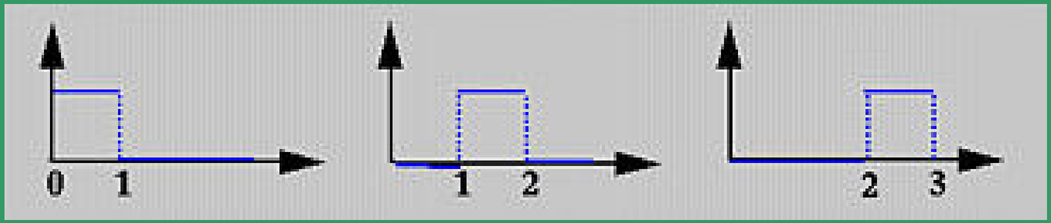


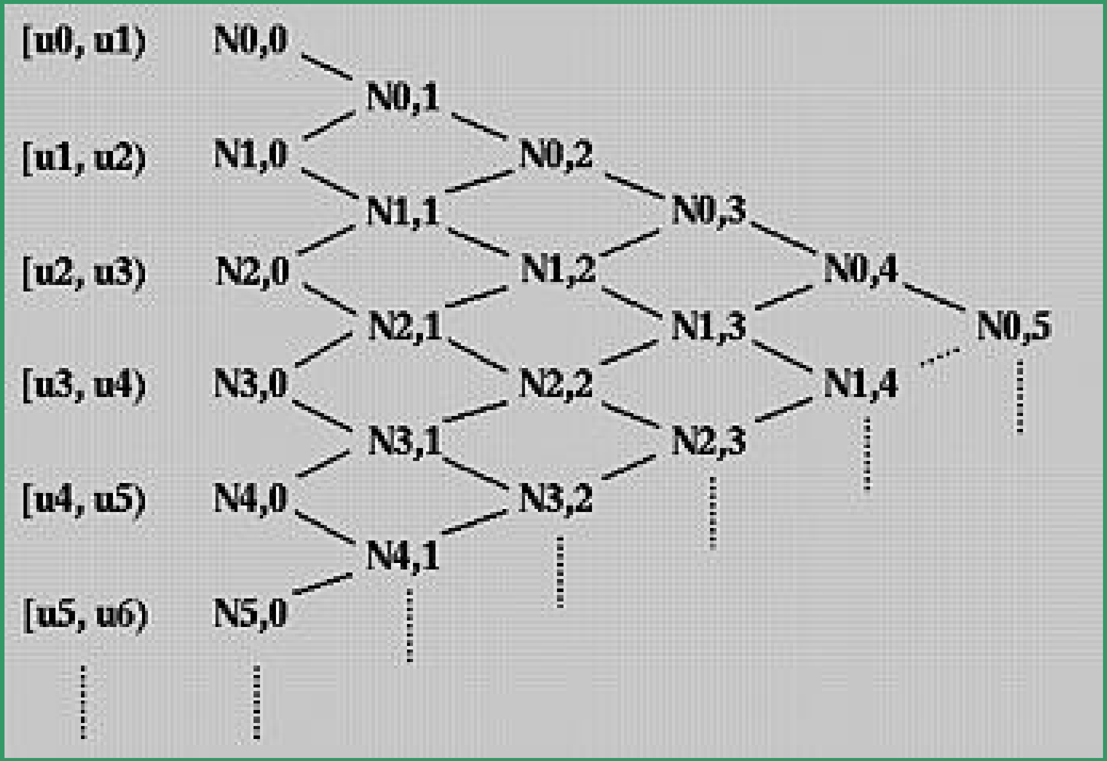
??

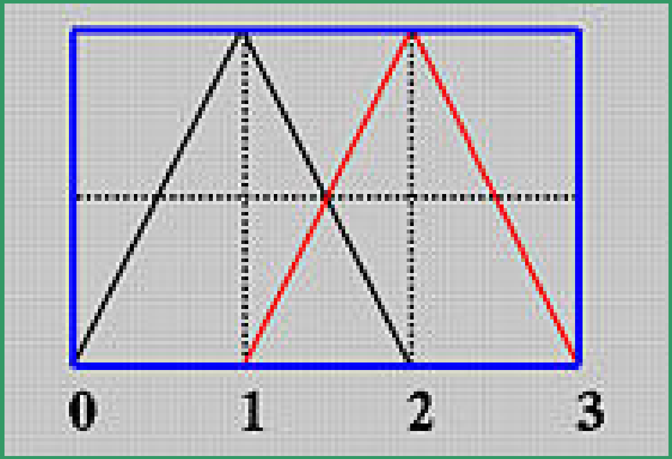
$$(n)B_{jn}(t) = (t-j)B_{j,n-1}(t) + (n+1+j-t)B_{j+1,n-1}(t)$$

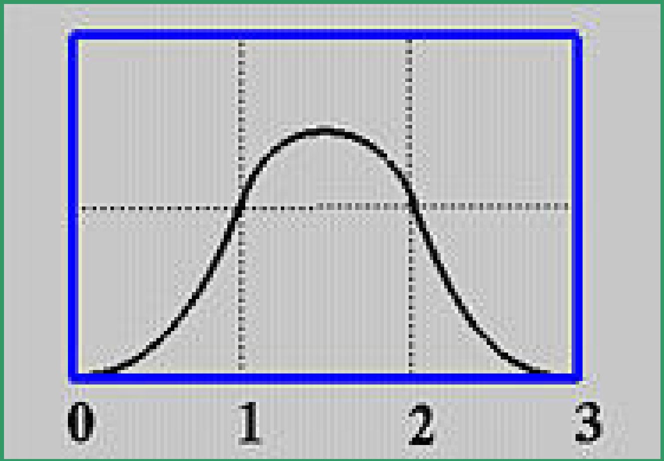
$$
g(t)=\begin{cases} a(t)=\dfrac{1}{2}t^2 & 0\le t\le 1\\[2mm] b(t)=\dfrac{3}{4}-\left(t-\dfrac{3}{2}\right)^2 & 1\le t\le 2\\[2mm] c(t)=\dfrac{1}{2}(3-t)^2 & 2\le t\le 3\end{cases}
$$











Joints: Values of functions at adjacent segments;

Knot – Values of t, where segments meet

Constant B-spline: $$b_{j,0}(t) = 1_{[t_j,t_{j+1}]} = \begin{cases} 1 & \text{if} \quad t_j \le t < t_{j+1} \\ 0 & \quad\text{otherwise} \end{cases}$$

Linear B-spline: $$b_{j,1}(t) = \begin{cases} \frac{t-t_j}{t_{j+1}-t_j} & \text{if} \quad t_j \le t < t_{j+1} \\ \frac{t_{j+2}-t}{t_{j+2}-t_{j+1}} & \text{if} \quad t_{j+1} \le t < t_{j+2} \\ 0 & \quad\text{otherwise} \end{cases}$$

Uniform quadratic B-spline (uniform knot vector):

$$b_{j,2}(t) = \begin{cases} \frac{1}{2}(t-t_j)^2 & t_j \le t \le t_{j+1} \\ -(t-t_{j+1})^2 + (t-t_{j+1}) + \frac{1}{2} & t_{j+1} \le t \le t_{j+2} \\ \frac{1}{2}(1-(t-t_{j+2}))^2 & t_{j+2} \le t \le t_{j+3} \\ 0 & \text{otherwise} \end{cases}$$

$\leftarrow =$

V = [1, 2, 3, 4, 5, 6];

Above, when reparameterized in the unit interval:

$$\mathbf{S}_i(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$

$$t \in [0,1], i = 1, 2 \ldots m-2$$

For the special case of the cubic B-spline ($k = 4$), the basis functions are

$$B_{i3}(s) = \begin{cases} \frac{1}{6}(s-i)^3 & \text{if } i \le s < i+1 \\ \frac{1}{6}[-3(s-i-1)^3 + 3(s-i-1)^2 + 3(s-i-1) + 1] & \text{if } i+1 \le s < i+2 \\ \frac{1}{6}[3(s-i-2)^3 - 6(s-i-2)^2 + 4] & \text{if } i+2 \le s < i+3 \\ \frac{1}{6}[1-(s-i-3)]^3 & \text{if } i+3 \le s < i+4 \\ 0 & \text{otherwise} \end{cases}$$

A Convenient Representation

Because of the local support property, we can rewrite the equation for a cubic B-spline as

$$p(s) = \frac{1}{6}\left[(1-(s-i))^3 p_{i-3} + [3(s-i)^3 - 6(s-i)^2 + 4]p_{i-2} + [-3(s-i)^3 + 3(s-i)^2 + 3(s-i) + 1]p_{i-1} + (s-i)^3 p_i\right],$$

where $i \le s < i+1$. A simila ... as

$$B_{0,3}(t) = (1-t)^3 / 6;$$

$$B_{1,3}(t) = (3.t^3 - 6t^2 + 4) / 6;$$

$$B_{2,3}(t) = (-3.t^3 + 3t^2 + 3t + 1) / 6;$$

$$B_{3,3}(t) = t^3 / 6.$$

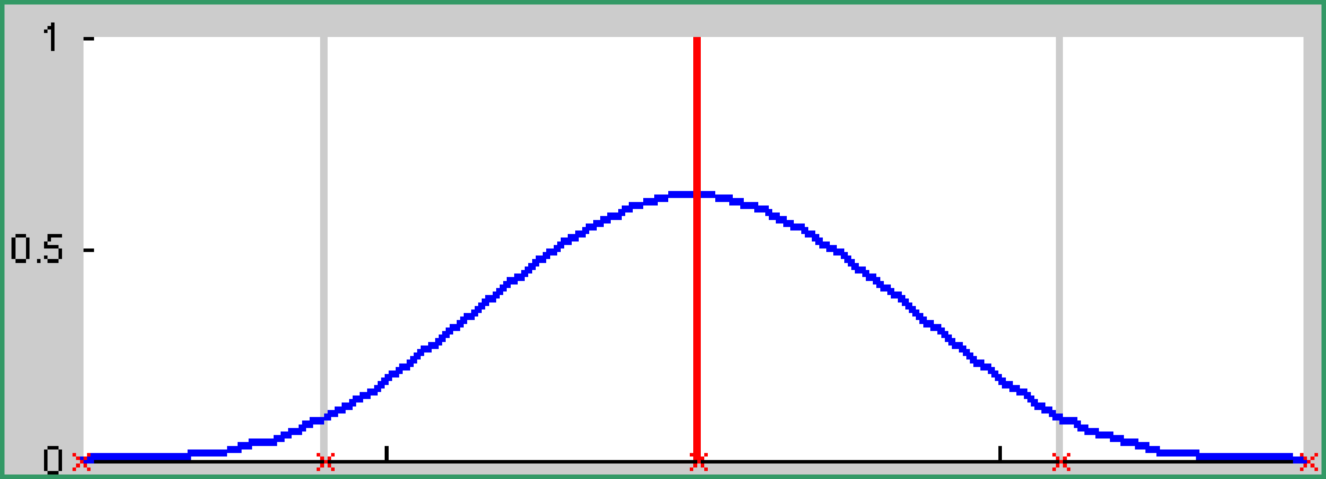
again where $i \le s < i+1$. In

$\mathbf{B}_i =$

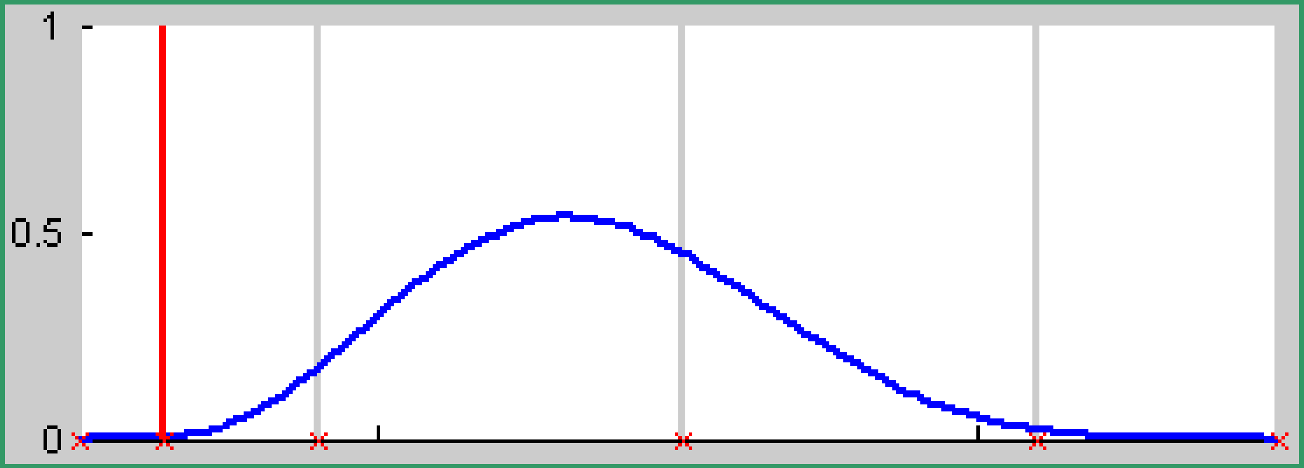
We can also include the placement matrix $\mathbf{G}_i$:

$$p(s) = [1 \quad s \quad s^2 \quad s^3]\mathbf{B}_i\mathbf{G}_i\mathbf{p},$$

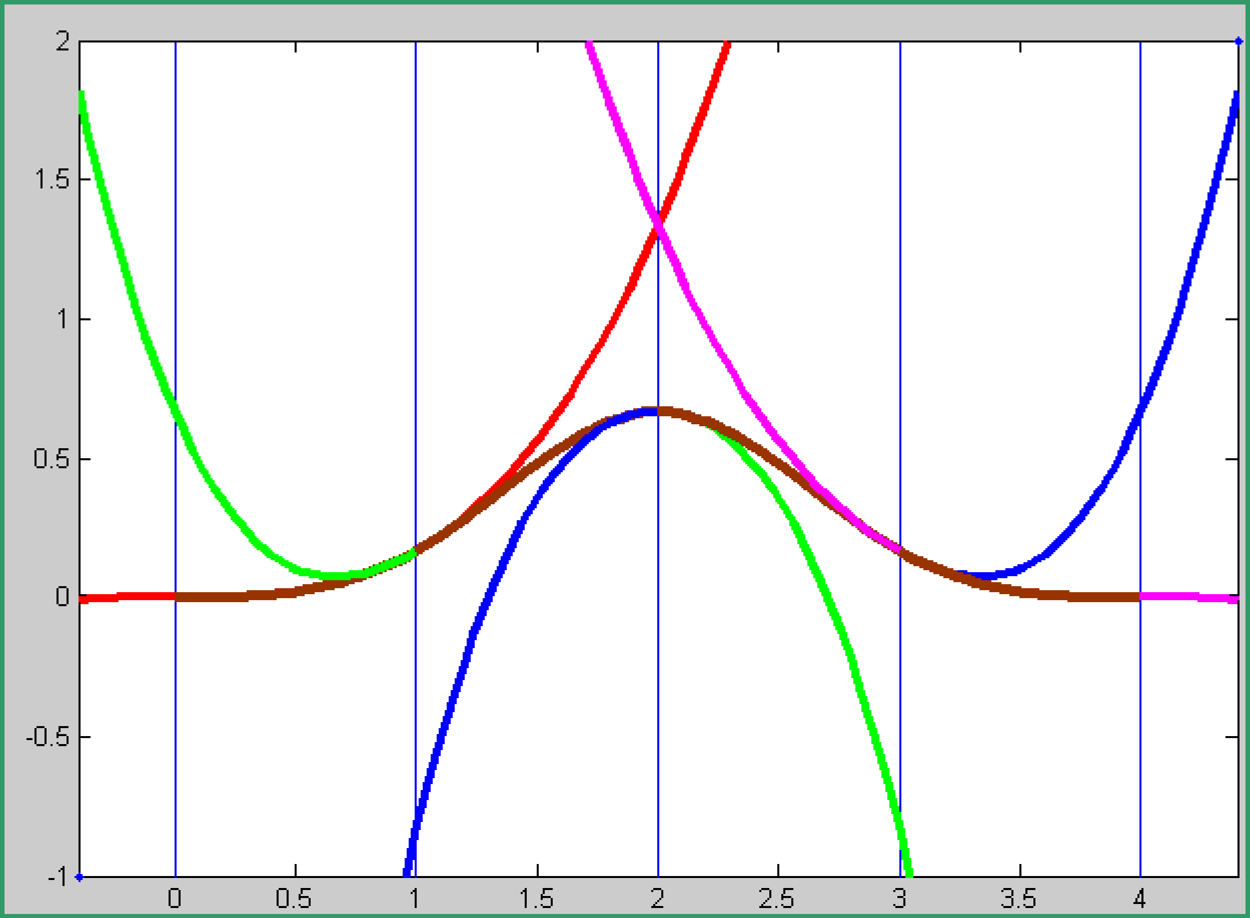
B-Spline Examples



Order 4, Degree 3, Knots = 5, Poly pieces = 4.

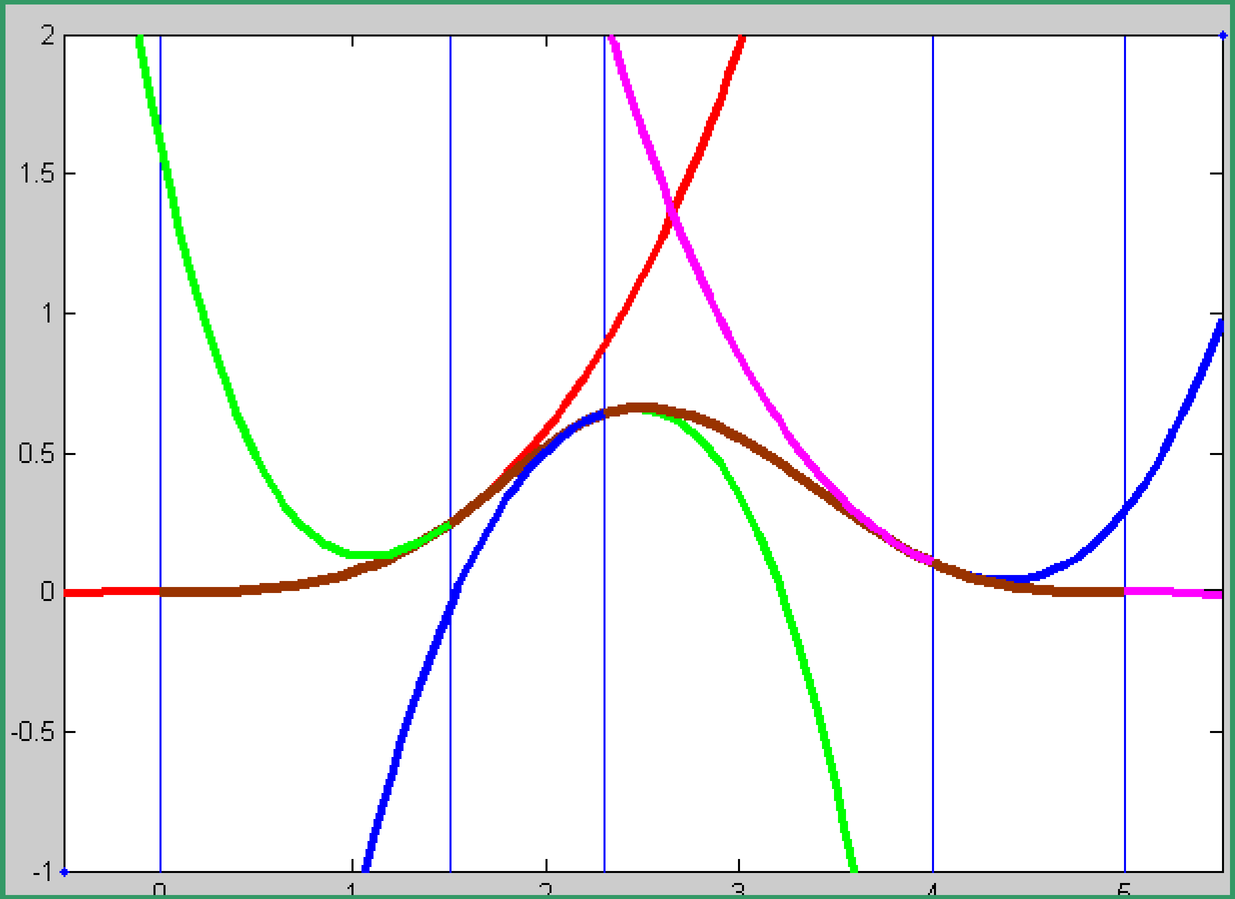


Order 5, Degree 4, Knots = 6, Poly pieces = 5.



A B-Spline of Order 4, and the Four Cubic Polynomials from which it is made.

Knot Sequence: [0 1 2 3 4]



A B-Spline of Order 4, and the Four Cubic Polynomials from which It Is Made

Knot Sequence: [0 1.5 2.3 4 5]

When the knots are equidistant we say the B-spline is uniform, otherwise we call it non-uniform.

NURBS: Non-uniform Regularized B-Splines

Uniform B-spline

When the B-spline is uniform, the basis B-splines for a given degree *n* are just shifted copies of each other. An alternative non-recursive definition for the *m−n-1* basis B-splines is:

$$b_{j,n}(t) = b_n(t - t_j), \qquad j = 0, \ldots, m - n - 2$$

with

$$b_n(t) := \frac{n+1}{n} \sum_{i=0}^{n+1} \omega_{i,n}(t - t_i)_+^n$$

and

$$\omega_{i,n} := \prod_{j=0, j\neq i}^{n+1} \frac{1}{t_j - t_i}$$

where

$$(t - t_i)_+^n := \begin{cases} (t - t_i)^n & \text{if } t \geq t_i \\ 0 & \text{if } t < t_i \end{cases}$$

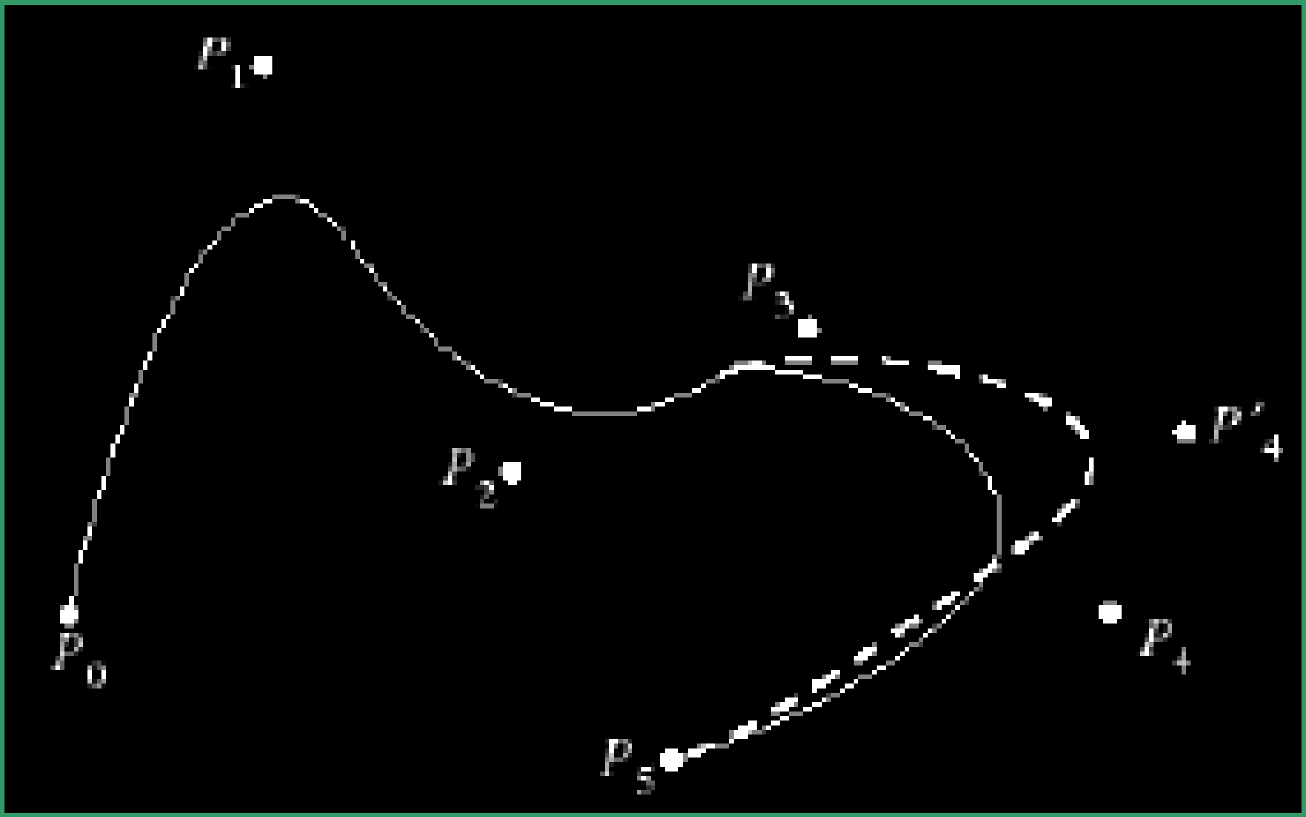
is the truncated power function.

When the number of Control points is the same as the order, the B-Spline degenerates into a Bézier curve.

$$t_0 = \ldots = t_n = 0$$

$$t_{n+1} = \ldots = t_{2n} = 1$$

The shape of the basis functions is determined by the position of the knots.



P_1
P_3
P'_4
P_2
P_0
P_4
P_5

For Bezier:

$$P(t)=\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ & & & 0 \\ & & & 0 \end{bmatrix}\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix};n=3.$$

$$B_{0,3}(t)=(1-t)^3/6;$$

$$B_{1,3}(t)=(3.t^3-6t^2+4)/6;$$

$$B_{2,3}(t)=(-3.t^3+3t^2+3t+1)/6;$$

$$B_{3,3}(t)=t^3/6.$$

For Cubic-splines:

$$P(t)=\begin{bmatrix} t^3 & & t & \end{bmatrix}\begin{bmatrix} \\ \\ \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_k \\ P_{k+1} \\ P_k' \\ P_{k+1}' \end{bmatrix}^T$$

For reparameterized Cubic B-splines, with uniform Knot vector:

$$\mathbf{S}_i(t)=\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\frac{1}{6}\begin{bmatrix} \boxed{} \end{bmatrix}\begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \\ \mathbf{p}_{i+2} \end{bmatrix}$$

$$P(t) = \sum_{i=1}^{4} B_i t^{i-1}; t_i \le t \le t_2.$$

$$P(u) = \sum_{k=0}^{3} g_k H_k(u)$$

$$P(t) = P_1(2t^3 - 3t^2 + 1) + P_2(-2t^3 + 3t^2)$$
$$+ P_1'(t^3 - 2t^2 + t) + P_2'(t^3 - t^2)$$

CUBIC SPLINES

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t); \quad 0 \le t \le 1$$

BEZIER CURVES

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}; \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$\mathbf{S}(t) = \sum_{i=0}^{m-n-1} \mathbf{P}_i b_{i,n}(t) \ , \ t \in [t_n, t_{m-n}]$$

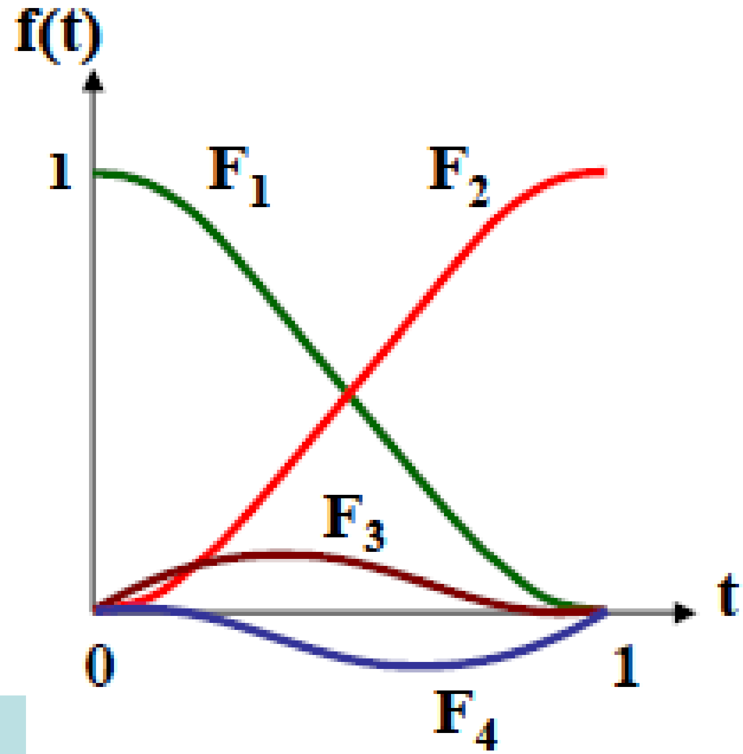
B-splines

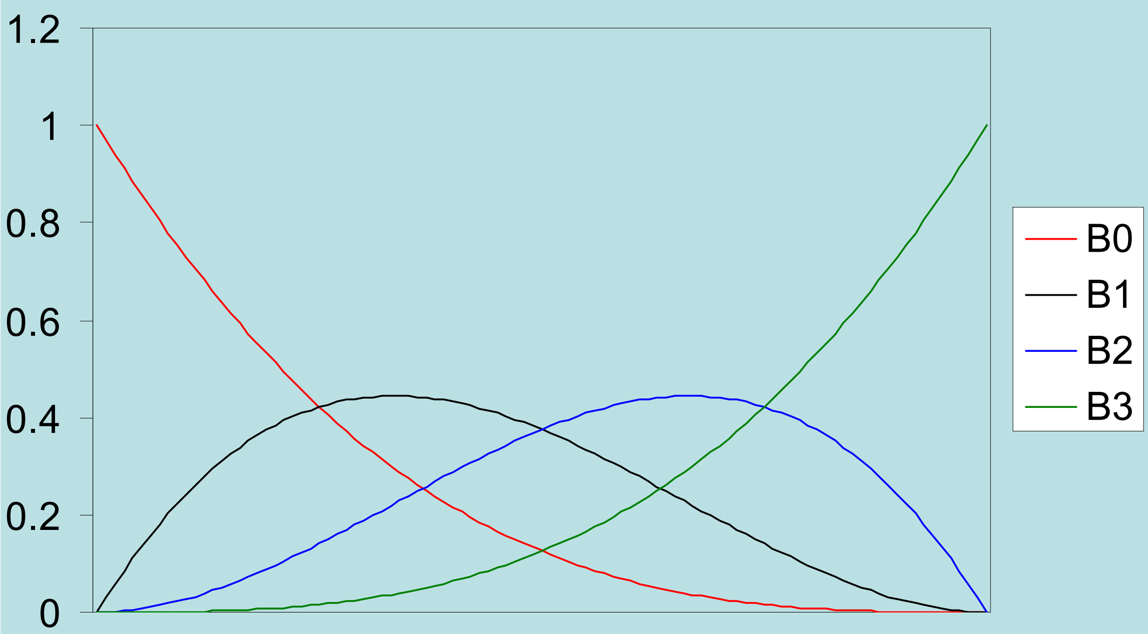
$$b_{j,0}(t) := \begin{cases} 1 & \text{if} \quad t_j \le t < t_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

j = 0,1,...,m-2

j = 0,1,...,m-n-2

$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t).$$


f(t)
1
F1
F2
F3
F4
t
0
1


1.2
1
0.8
0.6
0.4
0.2
0
B0
B1
B2
B3

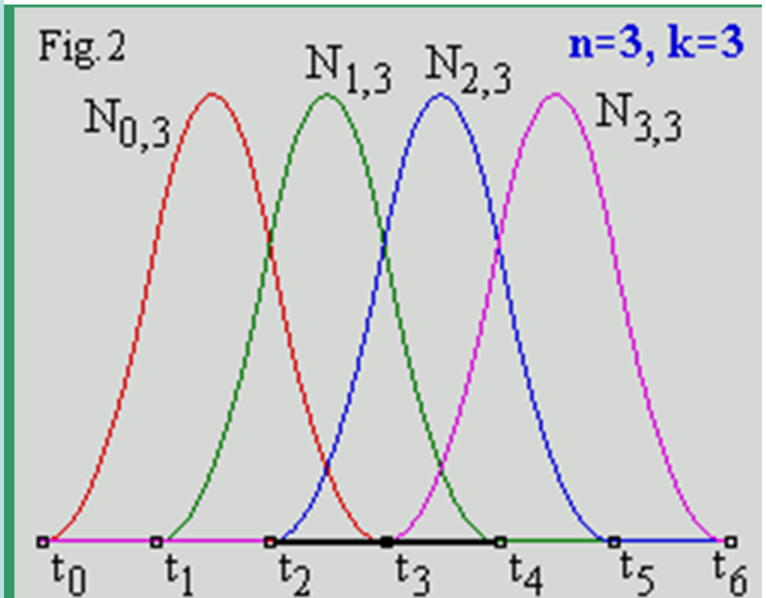
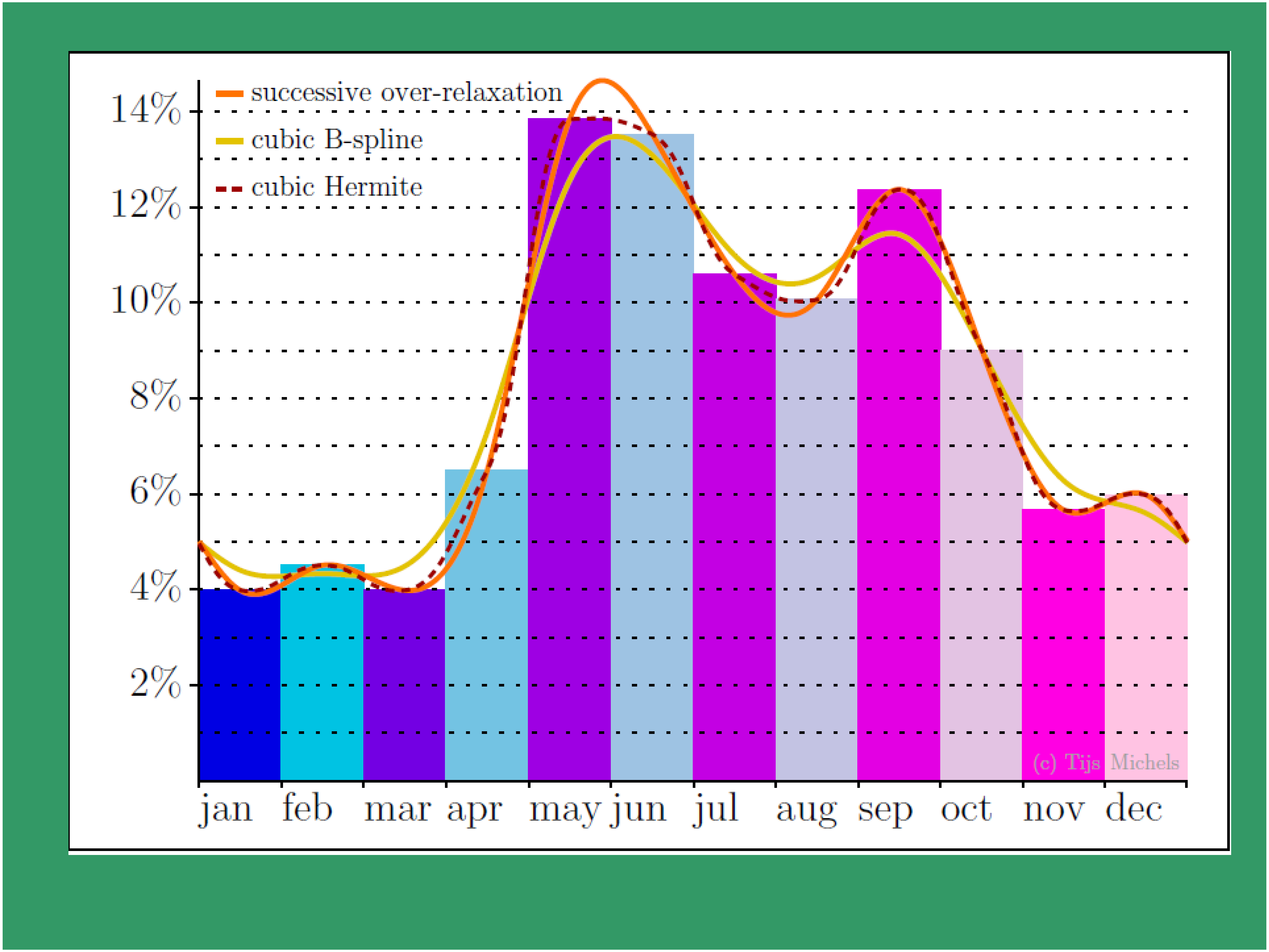
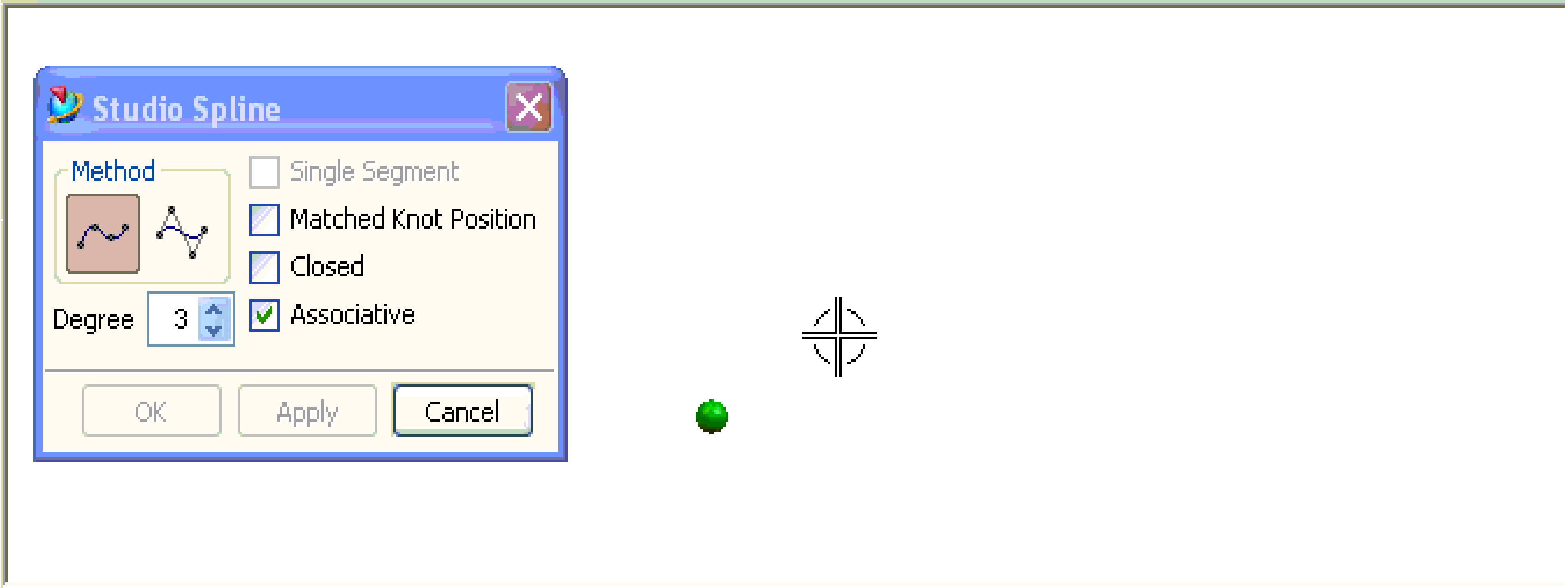


Fig. 2
n=3, k=3
N0,3
N1,3
N2,3
N3,3
t0
t1
t2
t3
t4
t5
t6



successive over-relaxation
cubic B-spline
cubic Hermite
14%
12%
10%
8%
6%
4%
2%
jan
feb
mar
apr
may
jun
jul
aug
sep
oct
nov
dec
(c) Tijs Michels


Studio Spline
Method
Single Segment
Matched Knot Position
Closed
Associative
Degree
3
OK
Apply
Cancel

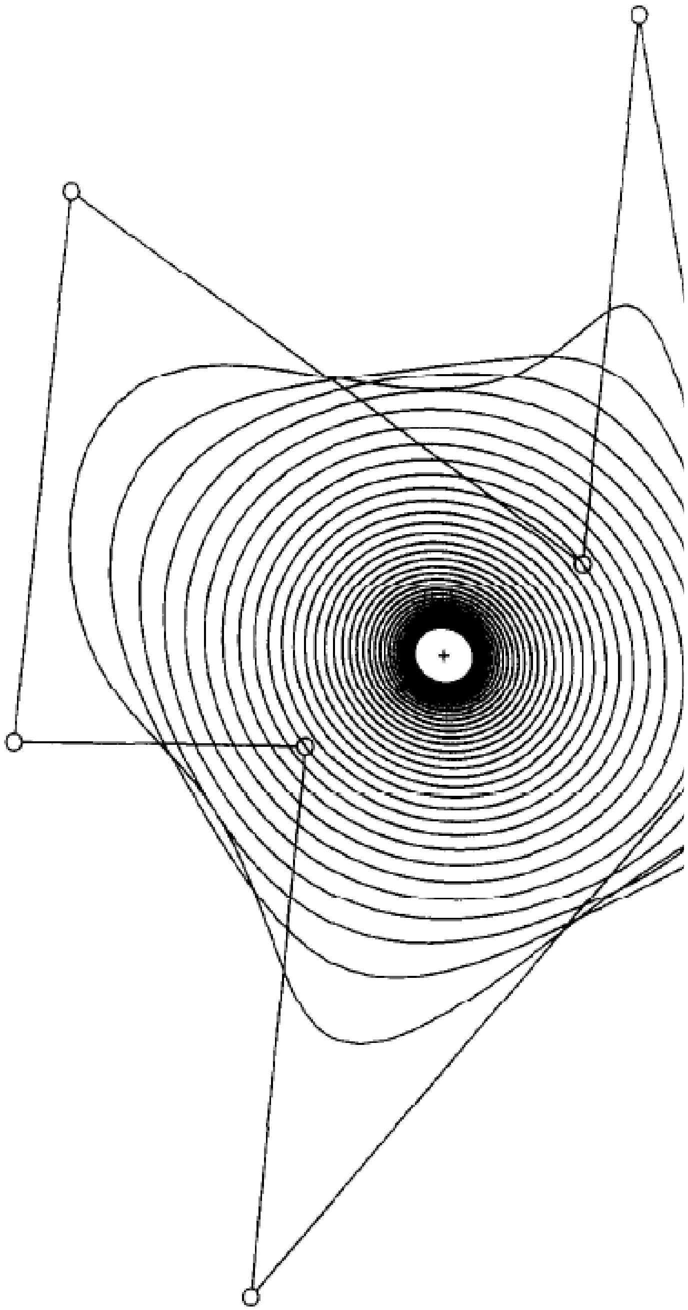


Figure 1. Spatial B-spline curves, only the curves of order k are drawn.

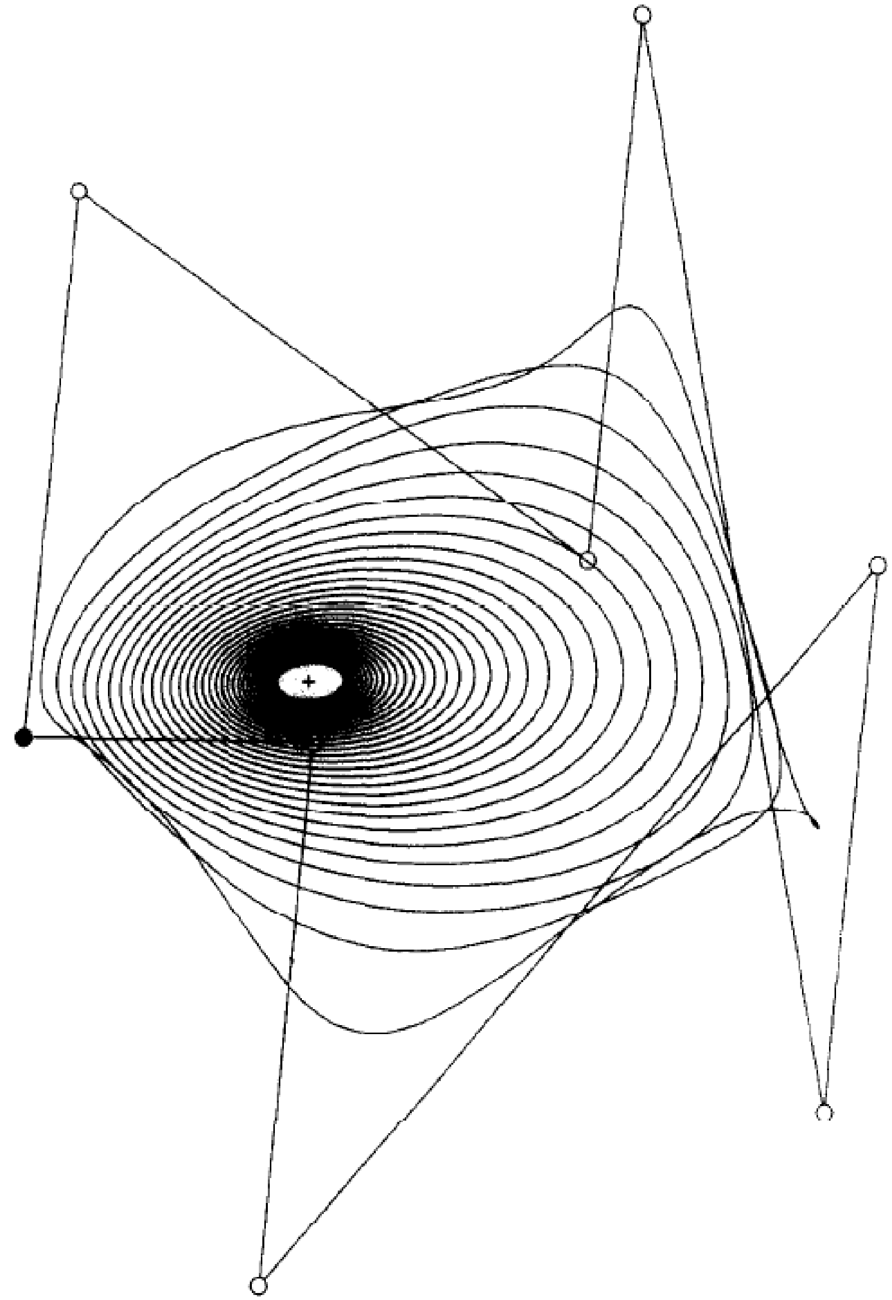


Figure 2. Spatial rational closed B-spline curves, only the curves of order $k = 2 + 3i$, $i = 0, 1, \ldots$ are drawn. The weight of the control point marked with filled dot is 4, while that of the rest is 1.

QUADRIC SURFACES

Some trivial examples:

SPHERE

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2;$$

$$x=r.\cos\phi.\cos\theta,\ -\pi/2\le\phi\le\pi/2$$

$$y=r.\cos\phi.\sin\theta,\ \ -\pi\le\phi\le\pi$$

$$z=r.\sin\phi.$$

ELLIPSOID

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1;$$

$$x = a.\cos\phi.\cos\theta, \quad -\pi/2 \le \phi \le \pi/2$$

$$y = b.\cos\phi.\sin\theta, \quad -\pi \le \phi \le \pi$$

$$z = c.\sin\phi.$$

TORUS

$$\left[r - \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}\,\right]^2 + \left(\frac{z}{c}\right)^2 = 1;$$

$$x = a.(r + \cos\phi).\cos\theta, \quad -\pi \le \phi \le \pi$$

$$y = b.(r + \cos\phi).\sin\theta, \quad -\pi \le \phi \le \pi$$

$$z = c.\sin\phi.$$

SUPERELLIPSOID

$$[(\frac{x}{a})^{2/s_2} + (\frac{y}{b})^{2/s_2}]^{s_2/s_1} + (\frac{z}{c})^{2/s_1} = 1;$$

$$x = a.\cos^{s_1}\phi.\cos^{s_2}\theta, \ -\pi/2 \le \phi \le \pi/2$$

$$y = b.\cos^{s_1}\phi.\sin^{s_1}\theta, \ \ -\pi \le \phi \le \pi$$

$$z = c.\sin s_1\phi.$$

SUPERQUADRICS:

$$(\alpha x)^n + (\beta y)^n + (\gamma z)^n = k$$

General expression of a Quadric Surface

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz$$
$$+ Gx + Hy + Jz + K = 0.$$

The above is a generalization of the general conic equation in 3-D. In matrix form, it is:

$$XSX^T = 0,$$

$$\Rightarrow \begin{bmatrix} x & y & z & 1 \end{bmatrix} (1/2) \begin{bmatrix} 2\mathrm{A} & D & F & G \\ D & 2B & E & H \\ F & E & 2C & J \\ G & H & J & 2K \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

Parametric forms of the quadric surfaces, are often used in computer graphics

Ellipsoid **:**

$$x = a\cos(\theta).\sin(\phi);\ \ 0 \le \theta \le 2\pi;$$
$$y = b\sin(\theta).\sin(\phi);\ \ 0 \le \phi \le 2\pi;$$
$$z = c\cos(\phi);$$

Elliptic Cone **:**

$$x = a\phi\cos(\theta);\ \ 0 \le \theta \le 2\pi$$
$$y = b\phi\sin(\theta);\ \ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = c\phi$$

Hyperbolic Paraboloid **:**

$$x = a\phi\cosh(\theta);\ \ -\pi \le \theta \le \pi$$
$$y = b\phi\sinh(\theta);\ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = \phi^2$$

Elliptic Paraboloid **:**

$$x = a\phi\cos(\theta);\ \ 0 \le \theta \le 2\pi$$
$$y = b\phi\sin(\theta);\ \ 0 \le \phi \le \phi_{\max}$$
$$z = \phi^2$$

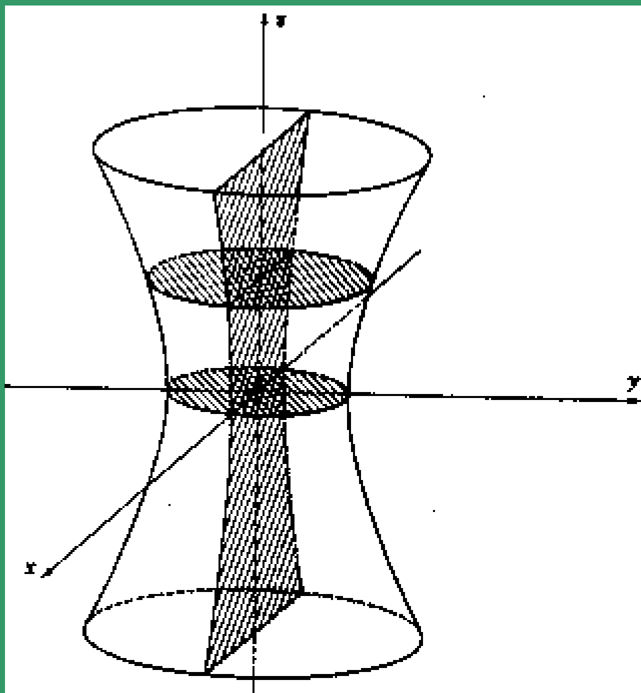
Hyperboloid:

$$x = a\cos(\theta)\cosh(\phi);\ \ 0 \le \theta \le 2\pi$$
$$y = b\sin(\theta)\sinh(\phi);\ \ -\pi \le \phi \le \pi$$
$$z = \sinh(\phi)$$

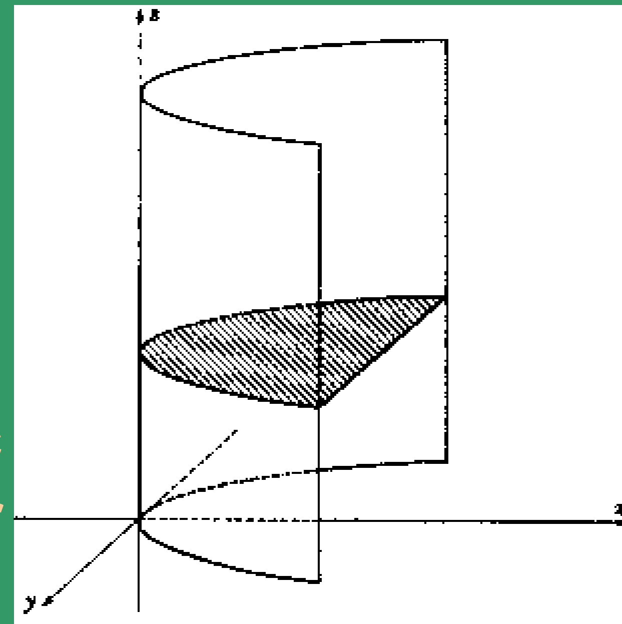
Parabolic Cylinder **:**

$$x = a\theta^2;\ \ 0 \le \theta \le \theta_{\max}$$
$$y = 2a\theta;\ \ \phi_{\min} \le \phi \le \phi_{\max}$$
$$z = \phi$$

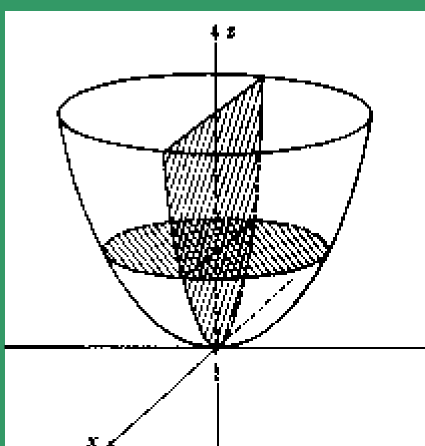
Some examples of Quadric Surfaces



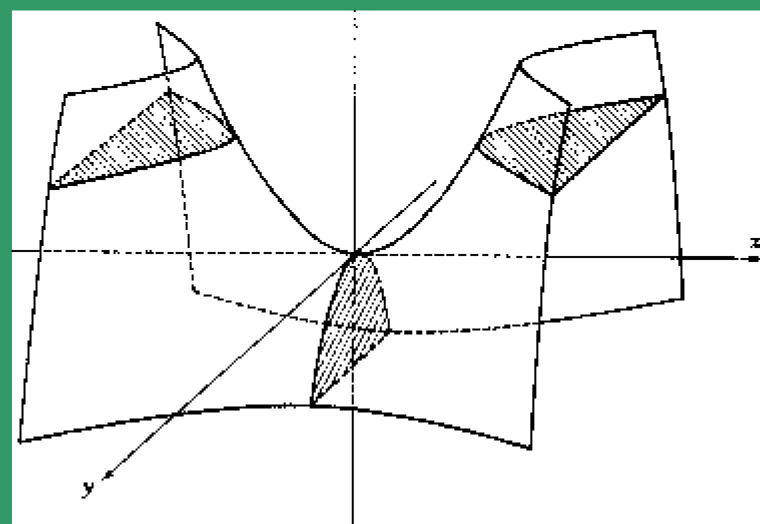
Hyperboloid



Parabolic
Cylinder



Elliptic
Paraboloid



Hyperbolic
Paraboloid

BEZIER Surfaces

- Degree of the surface in each parametric direction is one less than the number of defining polygon vertices in that direction
- Surface generally follows the shape of the defining polygon net
- Continuity of the surface in each parametric direction is two less than the number of defining polygon net
- Only the corner points of the defining polygon net and the surface are coincident
- The surface is contained within the convex hull of the defining polygon
- Surface is invariant under any affine transformation.

Equation of a parametric Bezier surface:

$$Q(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{i,j} J_{n,i}(u) K_{m,j}(w);$$

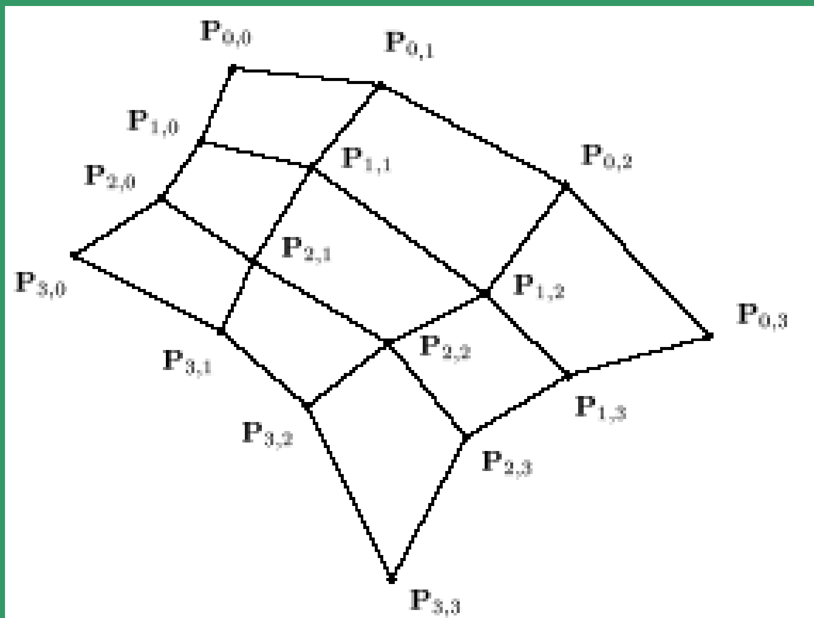
$$J_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i};$$

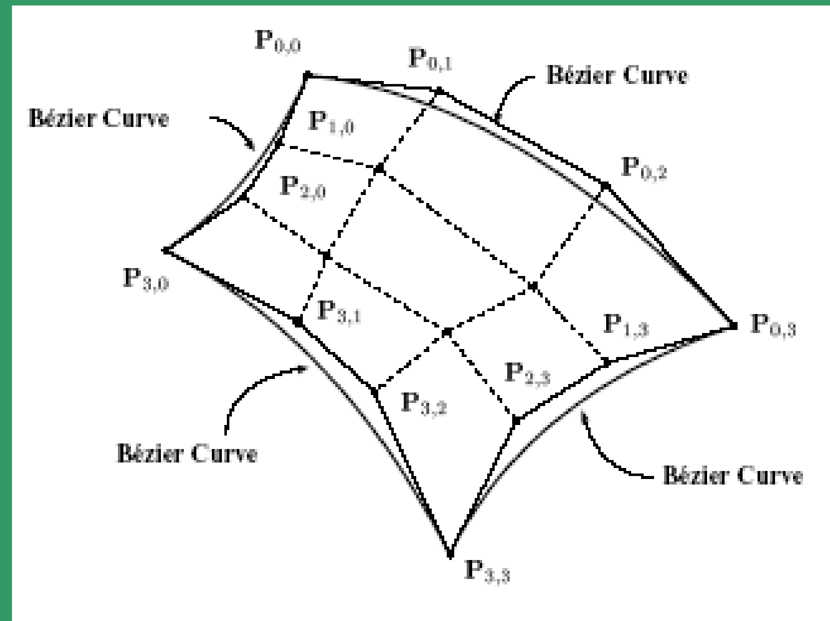
$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$K_{m,j}(w) = \binom{m}{j} w^j (1-w)^{m-j};$$

$$\binom{m}{j} = \frac{m!}{j!(m-j)!}$$

BEZIER Surfaces





$$Q(u,w) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} J_{n,i}(u) K_{m,j}(w)$$

$$= \sum_{i=0}^{n} \left[\sum_{j=0}^{m} P_{i,j} J_{n,i}(u) \right] K_{m,j}(w);$$

BEZIER Surface in matrix form:

$$Q(u,w) = U.N.B.M^T W;$$

where,

$$U = [u^n \quad u^{n-1} \quad . \quad . \quad 1],$$

$$W = [w^m \quad w^{m-1} \quad . \quad . \quad 1]^T,$$

$$B = \begin{bmatrix} B_{0,0} & . & . & B_{0,m} \\ . & . & . & . \\ . & . & . & . \\ B_{n,0} & . & . & B_{n,m} \end{bmatrix}$$

4x4 bicubic BEZIER Surface in matrix form:

$$Q(u,w) =$$

$$\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3} \\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,2} \\ B_{2,0} & B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,0} & B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix}$$

$$X \begin{bmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix};$$

Non-square 4x4 bicubic BEZIER Surface in matrix form:

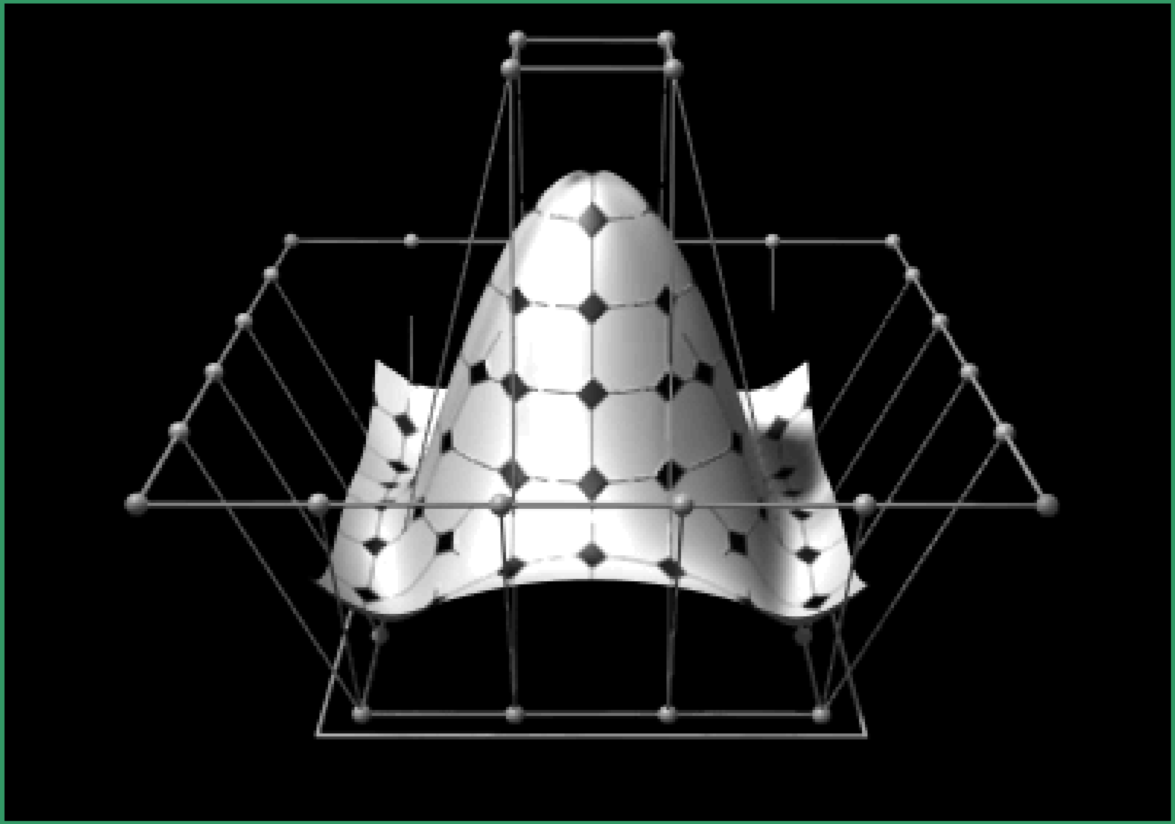
$$Q(u,w)=$$

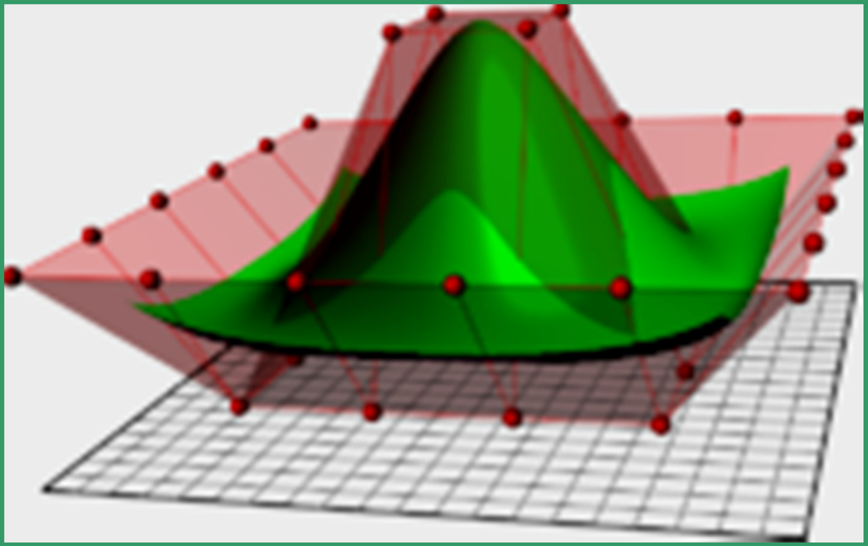
$$\begin{bmatrix} u^4 & u^3 & u^2 & u & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & -12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X\begin{bmatrix} B_{0,0} & B_{0,1} & B_{0,2} \\ B_{1,0} & B_{1,1} & B_{1,2} \\ B_{2,0} & B_{2,1} & B_{2,2} \\ B_{3,0} & B_{3,1} & B_{3,2} \\ B_{4,0} & B_{4,1} & B_{4,2} \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} w^2 \\ w \\ 1 \end{bmatrix};$$

NURBS

$$Q(u,v) = \frac{\sum_{i=0}^{M}\sum_{k=0}^{L} w_{i,j} P_{i,k} B_{i,m}(u) B_{k,n}(v)}{\sum_{i=0}^{M}\sum_{k=0}^{L} w_{i,j} B_{i,m}(u) B_{k,n}(v)}$$





End of Lectures on

CURVES
and SURFACE
REPRESENTATION