Chapter 02
Solution of Algebraic and Transcendental Equations
Chapter Outcomes:

After reading this chapter, you should be able to:

1. Know about the Error, exact, relative, and percentage error.
2. Relate the absolute, relative, and approximate error to the number of significant digits.
3. Know that there are two inherent sources of error in numerical methods – round-off and truncation error.
4. Know the difference between round-off and truncation error.
5. Know the concept of significant digits.
In this Lesson, we have discussed about the solution of equations, \( f(x) = 0 \)

Where, \( f(X) \) is \text{linear, nonlinear, algebraic} \ or \text{transcendental function}.

We get the solution of the equation \( f(x) = 0 \) by using

1. \textbf{Bisection method},
2. \textbf{Newton-Raphson method} and
3. \textbf{Method of false position}.

Those methods are established based on \textbf{Intermediate Value Theorem}. 
Statement of Intermediate Value Theorem:

If \( f(x) \) is continuous in the interval \((a, b)\) and if \( f(a) \) and \( f(b) \) are of opposite signs, then the equation \( f(x) = 0 \) will have at least one real root between \( a \) and \( b \).
◆ **Algebraic Equation:**
An algebraic equation is an equation that includes one or more variables such as

\[ x^2 + xy - z = 0 \]

◆ **Transcendental equation:**
An equation together with algebraic, trigonometrical, exponential or logarithmic function etc. is called transcendental equation such as

\[ e^x + 5 \sin x - 2x = 0 \]

◆ **Solution/root:**
A solution/root of an equation is the value of the variable or variables that satisfies the equation.
Iteration:

- Iteration is the repeated process of calculation until the desired result or approximate numerical value has come.

- Each repetition of the process is also called iteration and the result of one iteration is used as the starting point for the next iteration.
We are capable to find the root of algebraic or transcendental function by using following methods:

1. Bisection method
2. Newton Rapshon method (Newton’s Iteration method)
3. Iteration method (Method of successive approximation/Fixed point Iteration Method)
4. Regular-Falsi method (The method of False position)
5. The secant method
*** You have to know, how to use calculator for solving the mathematical problem

*** No. 01 : To determine the value of the trigonometrical function f(x), we have to change our calculator in radian mode.
Use of Calculator

*** You have to know, how to use calculator for solving the mathematical problem.
Bisection Method
Procedure of Bisection Method:

Step 1: Given function, $f(x)$

Let, $x^3 + x^2 - 1 = 0$

Given, $f(x) = x^3 + x^2 - 1$
\[ f(x) = x^3 + x^2 - 1 \]

**Step 2:** Choose two real numbers \( a \) and \( b \) such that,
\[
\text{Find the Interval between } (a, b) \text{ in the step 2}
\]
\[
\begin{align*}
\text{For, } a &= x = -1, \\
\therefore f(a) &= a^3 + a^2 - 1 \\
&= (-1)^3 + (-1)^2 - 1 = -1 < 0
\end{align*}
\]
\[
\begin{align*}
\text{For, } b &= x = 2, \\
\therefore f(b) &= b^3 + b^2 - 1 \\
&= 2^3 + 2^2 - 1 = 10 > 0
\end{align*}
\]
\[
\begin{align*}
\therefore f(0) &= 0^3 + 0^2 - 1 = -1 \\
\therefore f(1) &= 1^3 + 1^2 - 1 = +1
\end{align*}
\]
\[
\therefore f(a) \times f(b) = f(0) \times f(1) = (-1) \times 1 = -1 < 0
\]

Since \( f(a) = f(0) \) is negative and \( f(b) = f(1) \) is positive,
So at least one real root lies between 0 and 1.
Step 3: Find the midpoint of a and b say, \(c\).

\[
\therefore c = \frac{a + b}{2}
\]

C is the root of the given function if \(f(c) = 0\); else follow the next step.

Step 4: Find \(f(c)\).

\[
a=0, b=1
\]

\[
root, c = \frac{a + b}{2} = \frac{0 + 1}{2} = \frac{1}{2} = 0.5
\]

\[
f(x) = x^3 + x^2 - 1
\]

\[
\therefore f(c) = c^3 + c^2 - 1
\]

\[
= (0.5)^3 + (0.5)^2 - 1
\]

\[
= 0.125 + 0.25 - 1
\]

\[
= -0.625
\]
Step 4: Case I: If $f(c)$ is negative, Then $b = c$  
Case II: If $f(c)$ is positive, Then $a = c$

Case I: If $f(c)$ is negative, Then $b = c$  
Case II: If $f(c)$ is positive, Then $a = c$

<table>
<thead>
<tr>
<th>No of Iterations</th>
<th>Value of $a$ [ (+) ve]</th>
<th>Value of $b$ [ (-) ve]</th>
<th>$c = \frac{a+b}{2}$</th>
<th>$f(c)$</th>
<th>Sign of $f(c)$</th>
<th>$b = c$</th>
<th>$a = c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>1</td>
<td>0</td>
<td>0.5</td>
<td>-0.625</td>
<td>&lt;0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>02</td>
<td>1</td>
<td>0.5</td>
<td>0.75</td>
<td>-0.015</td>
<td>&lt;0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>03</td>
<td>1</td>
<td>0.75</td>
<td>0.875</td>
<td>0.43555</td>
<td>&gt;0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>04</td>
<td>0.875</td>
<td>0.75</td>
<td>0.8125</td>
<td>0.19653</td>
<td>&gt;0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>05</td>
<td>0.8125</td>
<td>0.75</td>
<td>0.78125</td>
<td>0.0871</td>
<td>&gt;0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Step 5: Repeat steps 2, 3, 4 until the last two iterations are equal or the difference between the last two iterations are near to zero.
\[ f(c) = c^3 + c^2 - 1 \]  

Steps 3, 4, 5

<table>
<thead>
<tr>
<th>No of Iterations</th>
<th>Value of a [+ve]</th>
<th>Value of b [-ve]</th>
<th>( c = \frac{a+b}{2} )</th>
<th>( f(c) )</th>
<th>Sign of ( f(c) )</th>
<th>Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>05</td>
<td>0.8125</td>
<td>0.75</td>
<td>0.78125</td>
<td>0.0871</td>
<td>&gt;0</td>
<td>a = c</td>
</tr>
<tr>
<td>06</td>
<td>0.78125</td>
<td>0.75</td>
<td>0.765625</td>
<td>0.03497</td>
<td>&gt;0</td>
<td>a = c</td>
</tr>
<tr>
<td>07</td>
<td>0.765625</td>
<td>0.75</td>
<td>0.7578125</td>
<td>0.00947</td>
<td>&gt;0</td>
<td>a = c</td>
</tr>
<tr>
<td>08</td>
<td>0.7578125</td>
<td>0.75</td>
<td>0.75390625</td>
<td>-0.0031</td>
<td>&lt;0</td>
<td>b = c</td>
</tr>
<tr>
<td>09</td>
<td>0.7578125</td>
<td>0.75390625</td>
<td>0.755859375</td>
<td>0.0035</td>
<td>&gt;0</td>
<td>a = c</td>
</tr>
<tr>
<td>10</td>
<td>0.755859375</td>
<td>0.75390625</td>
<td>0.75488281255</td>
<td>1.65...</td>
<td>&gt;0</td>
<td>a = c</td>
</tr>
<tr>
<td>11</td>
<td>0.75488281255</td>
<td>0.75390625</td>
<td>0.754394531275</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The difference between the last two iterations are near to zero.
It is evident that from the above table, the difference between last two successive iterative values of $x$ is

$$|0.75488281255 - 0.754394531275| \approx 0.0005 \text{ or near to zero.}$$

which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is 0.754.
Problem 1:

Find the root of the equation \( xe^x = 1 \) by using Bisection Method correct up to three decimal places on the interval \((0,1)\).

Solution: \( f(x) = xe^x - 1 \)

Let, \( a = 0 \), \( b = 1 \) then,

\[
\begin{align*}
f(a) &= a.e^a - 1 \\
\therefore f(0) &= 0.e^0 - 1 = -1 < 0
\end{align*}
\]

Again, \( f(b) = b.e^b - 1 \)

\[
\begin{align*}
f(1) &= 1.e^1 - 1 = 1.7182 > 0
\end{align*}
\]

Since \( f(0) \) and \( f(1) \) are of opposite sign so at least one real root lies between 0 and 1.
\[ \text{mid point, } c = \frac{a + b}{2} \]

\[ = \frac{0 + 1}{2} = \frac{1}{2} = 0.5 \]

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.
<table>
<thead>
<tr>
<th>No. of iterations</th>
<th>Value of a [ (+) ve]</th>
<th>Value of a [ (-) ve]</th>
<th>$c = \frac{a+b}{2}$</th>
<th>$f(c) = ce^c - 1$</th>
<th>Sign of f (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>1</td>
<td>0</td>
<td>0.5</td>
<td>-0.1756</td>
<td>&lt;0</td>
</tr>
<tr>
<td>02</td>
<td>1</td>
<td>0.5</td>
<td>0.75</td>
<td>0.5877</td>
<td>&gt;0</td>
</tr>
<tr>
<td>03</td>
<td>0.75</td>
<td>0.5</td>
<td>0.625</td>
<td>0.1676</td>
<td>&gt;0</td>
</tr>
<tr>
<td>04</td>
<td>0.625</td>
<td>0.5</td>
<td>0.5625</td>
<td>-0.0127</td>
<td>&lt;0</td>
</tr>
<tr>
<td>05</td>
<td>0.625</td>
<td>0.5625</td>
<td>0.59375</td>
<td>0.0751</td>
<td>&gt;0</td>
</tr>
<tr>
<td>06</td>
<td>0.59375</td>
<td>0.5625</td>
<td>0.578125</td>
<td>0.0306</td>
<td>&gt;0</td>
</tr>
<tr>
<td>07</td>
<td>0.578125</td>
<td>0.5625</td>
<td>0.5703125</td>
<td>0.00877</td>
<td>&gt;0</td>
</tr>
<tr>
<td>08</td>
<td>0.5703125</td>
<td>0.5625</td>
<td>0.56640625</td>
<td>-0.0023</td>
<td>&lt;0</td>
</tr>
<tr>
<td>09</td>
<td>0.5703125</td>
<td>0.56640625</td>
<td>0.5683594</td>
<td>0.00336</td>
<td>&gt;0</td>
</tr>
<tr>
<td>10</td>
<td>0.5683594</td>
<td>0.56640625</td>
<td>0.5673828</td>
<td>0.000662</td>
<td>&gt;0</td>
</tr>
</tbody>
</table>
It is evident that from the above table, the difference between last two successive iterative values of $x$ is

$$|0.5683594 - 0.5673828| \approx 0.001$$

which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is 0.567
**Problem 2:** Find the root of the equation \(4 \sin x - e^x = 0\) by using Bisection method correct up to four decimal places.

**Solution:** Consider that, \(f(x) = 4 \sin x - e^x\)

Her, let \(a = 0\), \(b = 1\) then,

\[
f(a) = 4 \sin a - e^a
\]

\[
f(0) = 4 \sin 0 - e^0
\]

\[
= 0 - 1
\]

\[
= -1.000 < 0
\]

Again, \(f(b) = 4 \sin b - e^b\)

\[
= 3.3658839392 - 2.7182818285
\]

\[
= 0.6476021107 > 0
\]

Since \(f(0)\) and \(f(1)\) are of opposite sign so at least one real root lies between 0 and 1.
\[ c = \frac{a + b}{2} \]

\[ = \frac{0 + 1}{2} = \frac{1}{2} = 0.5 \]

Number of iterations for bisection method is given in the following table in arranged way for determining the approximate value of the desired root of the given equation.
<table>
<thead>
<tr>
<th>No of Iterations</th>
<th>Value of a [ (+) ve]</th>
<th>Value of b [ (-) ve]</th>
<th>( c = \frac{a+b}{2} )</th>
<th>( f(c) )</th>
<th>Sign of ( f(c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>1</td>
<td>0</td>
<td>0.5</td>
<td>0.268</td>
<td>&gt;0</td>
</tr>
<tr>
<td>02</td>
<td>0.5</td>
<td>0</td>
<td>0.25</td>
<td>-0.294</td>
<td>&lt;0</td>
</tr>
<tr>
<td>03</td>
<td>0.5</td>
<td>0.25</td>
<td>0.375</td>
<td>0.0101</td>
<td>&gt;0</td>
</tr>
<tr>
<td>04</td>
<td>0.375</td>
<td>0.25</td>
<td>0.3125</td>
<td>-0.1371</td>
<td>&lt;0</td>
</tr>
<tr>
<td>05</td>
<td>0.375</td>
<td>0.3125</td>
<td>0.34375</td>
<td>-0.0621</td>
<td>&lt;0</td>
</tr>
<tr>
<td>06</td>
<td>0.375</td>
<td>0.34375</td>
<td>0.359375</td>
<td>-0.0256</td>
<td>&lt;0</td>
</tr>
<tr>
<td>07</td>
<td>0.375</td>
<td>0.359375</td>
<td>0.3671875</td>
<td>-0.0077</td>
<td>&lt;0</td>
</tr>
<tr>
<td>08</td>
<td>0.375</td>
<td>0.3671875</td>
<td>0.3710937</td>
<td>-0.00122</td>
<td>&lt;0</td>
</tr>
<tr>
<td>09</td>
<td>0.375</td>
<td>0.3710937</td>
<td>0.373046</td>
<td>0.00566</td>
<td>&gt;0</td>
</tr>
<tr>
<td>10</td>
<td>0.373046</td>
<td>0.3710937</td>
<td>0.372070</td>
<td>-0.00344</td>
<td>&lt;0</td>
</tr>
<tr>
<td>No of Iterations</td>
<td>Value of a [ (+) ve]</td>
<td>Value of b [ (-) ve]</td>
<td>$c = \frac{a+b}{2}$</td>
<td>f (c)</td>
<td>Sign of f (c)</td>
</tr>
<tr>
<td>------------------</td>
<td>----------------------</td>
<td>----------------------</td>
<td>---------------------</td>
<td>--------</td>
<td>---------------</td>
</tr>
<tr>
<td>10</td>
<td>0.373046</td>
<td>0.3710937</td>
<td>0.372070</td>
<td>-0.00344</td>
<td>&lt;0</td>
</tr>
<tr>
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<td>0.372070</td>
<td>0.372558</td>
<td>0.00455</td>
<td>&gt;0</td>
</tr>
<tr>
<td>12</td>
<td>0.372558</td>
<td>0.372070</td>
<td>0.372279</td>
<td>0.0039</td>
<td>&gt;0</td>
</tr>
<tr>
<td>13</td>
<td>0.372279</td>
<td>0.372070</td>
<td>0.372174</td>
<td>0.0036</td>
<td>&gt;0</td>
</tr>
<tr>
<td>14</td>
<td>0.372174</td>
<td>0.372070</td>
<td>0.372122</td>
<td>0.0036</td>
<td>&gt;0</td>
</tr>
</tbody>
</table>
It is evident that from the above table, the difference between last two successive iterative values of x is

$$|0.372174 - 0.372122| = 0.0000520000 \approx 0.00001$$

which the accuracy condition for the solution exact. So, the required root of the given equation up to the three decimal places is 0.3721
**Algorithm of Bisection Method:**

<table>
<thead>
<tr>
<th>Steps</th>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>Define $f(x)$</td>
</tr>
<tr>
<td>02</td>
<td>Read a ‘The lower bound of the desired roots’</td>
</tr>
<tr>
<td>03</td>
<td>Read b ‘The upper bound of the desired roots’</td>
</tr>
<tr>
<td>04</td>
<td>Set $k = 1$</td>
</tr>
<tr>
<td>05</td>
<td>Calculate $x_k = \frac{a+b}{2}$</td>
</tr>
<tr>
<td>06</td>
<td>Calculate $f_k = f(x_k)$</td>
</tr>
<tr>
<td>07</td>
<td>Print $k, x_k, f_k$</td>
</tr>
</tbody>
</table>
| 08    | If $|x_k - x_{k-1}| \approx 0.0001$ then  
|      | GOTO Step 11 |
|      | elseif  
|      | $f(a).f_k < 0$ then $b = x_k$. |
|      | Else  
|      | $f(b).f_k < 0$ then $a = x_k$. |
|      | Endif |
| 09    | Set $k = k + 1$ |
| 10    | GOTO Step 05 |
| 11    | Print ‘Required root, $x_k$’ |
| 12    | STOP |
Find the root of the following equation by using Bisection method correct up to four decimal places:

1. $2^x - 5x + 2 = 0$
2. $2x + \cos x - 3 = 0$
3. $x^2 - 4x - 10 = 0$
4. $2x = 1 + \sin x$
5. $\cos x - x e^x = 0$
6. $e^x \tan x = 1$
7. $\cos(x) - \log(x) = 0$