Rules of Inferences
Discrete Mathematics — CSE 131

Outline

Outline

Example: Existence of Superman

If Superman were able and willing to prevent evil, then he would so. If Superman were unable to prevent evil, then he would be impotent; if he were unwilling to prevent evil, then he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

Is this argument valid ?

Outline

Definitions

By an **argument**, we mean a sequence of statements that ends with a **conclusion**.

The **conclusion** is the last statement of the argument.

The **premises** are the statements of the argument preceding the conclusion.

By a **valid argument**, we mean that the conclusion must follow from the truth of the premises.

Rule of Inference

Some tautologies are **rules of inference**. The general form of a rule of inference is

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow c$$

where

p_i are the **premises**

and

c is the **conclusion**.

Notation

A rule of inference is written as

$$\begin{array}{rl} & p_1 \\ & p_2 \\ & \vdots \\ & p_n \\ \hline \therefore & c \end{array}$$

where the symbol $\therefore$ denotes "**therefore**". Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion.

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modus ponens

The rule of inference

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

is denoted the **law of detachment** or *modus ponens* (Latin for *mode that affirms*). If a conditional statement and the hypothesis of the conditional statement are both true, therefore the conclusion must also be true.

The basis of the *modus ponens* is the tautology

$$((p \rightarrow q) \wedge p) \rightarrow q.$$

modus ponens

p	q	$p \to q$	$p \land (p \to q)$	$(p \land (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Example of *modus ponens*

If it rains, then it is cloudy.
It rains.
Therefore, it is cloudy.

r is the proposition "it rains."
c is the proposition "it is cloudy."

$$\begin{array}{rl} & r \longrightarrow c \\ & r \\ \hline \therefore & c \end{array}$$

modus tollens

The rule of inference

$$\begin{array}{rl} & p \rightarrow q \\ & \neg q \\ \hline \therefore & \neg p \end{array}$$

is denoted the *modus tollens* (Latin for *mode that denies*). This rule of inference is based on the contrapositive. The basis of the *modus ponens* is the tautology

$$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p.$$

modus tollens

p	q	$p \rightarrow q$	$\neg q$	$(p \rightarrow q) \wedge \neg q$	$\neg p$	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Example of *modus tollens*

If it rains, then it is cloudy.
It is not cloudy.
Therefore, it is not the case that it rains.

r is the proposition "it rains."
c is the proposition "it is cloudy."

$$\begin{array}{rl} & r \longrightarrow c \\ & \neg c \\ \hline \therefore & \neg r \end{array}$$

The Addition

The rule of inference

$$\frac{p}{\therefore \; p \vee q}$$

is the rule of **addition**.

This rule comes from the tautology

$$p \rightarrow (p \vee q).$$

The Simplification

The rule of inference

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

is the rule of **simplification**.

This rule comes from the tautology

$$(p \wedge q) \rightarrow p.$$

The Hypothetical Syllogism

The rule of inference

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

is the rule of **hypothetical syllogism** (syllogism means “argument made of three propositions where the last one, the conclusion, is necessarily true if the two firsts, the hypotheses, are true”).

This rule comes from the tautology

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r).$$

The Disjunctive Syllogism

The rule of inference

$$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

is the rule of **disjunctive syllogism**.

This rule comes from the tautology

$$((p \vee q) \wedge \neg p) \rightarrow q.$$

The Conjunction

The rule of inference

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

is the rule of **conjunction**.

This rule comes from the tautology

$$((p) \wedge (q)) \rightarrow (p \wedge q).$$

The Resolution

The rule of inference

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

is the rule of **resolution**.

This rule comes from the tautology

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r).$$

Outline

Fallacies

Fallacies are incorrect arguments.

Fallacies resemble rules of inference but are based on contingencies rather than tautologies.

The Fallacy of Affirming the Conclusion

The wrong "rule of inference"

$$\begin{array}{rl} & p \rightarrow q \\ & q \\ \hline \therefore & p \end{array}$$

is denoted the **fallacy of affirming the conclusion**.

The basis of this fallacy is the contingency

$$(q \wedge (p \rightarrow q)) \rightarrow p$$

that is a misuse of the *modus ponens* and is not a tautology.

Fallacy of Affirming the Conclusion

p	q	$p \rightarrow q$	$q \wedge (p \rightarrow q)$	$(q \wedge (p \rightarrow q)) \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Example of the Fallacy of Affirming the Conclusion

If it rains, then it is cloudy.
It is cloudy.
Therefore, it rains (wrong).

r is the proposition "it rains."
c is the proposition "it is cloudy."

$$\begin{array}{l} r \longrightarrow c \\ c \\ \hline \therefore \; r \text{ (wrong)} \end{array}$$

The Fallacy of Denying the Hypothesis

The wrong "rule of inference"

$$\begin{array}{rl} & p \rightarrow q \\ & \neg p \\ \hline \therefore & \neg q \end{array}$$

is denoted the **fallacy of denying the hypothesis**.

The basis of this fallacy is the contingency

$$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q$$

that is a misuse of the *modus tollens* and is not a tautology.

Fallacy of Denying the Hypothesis

p	q	$p \to q$	$\neg p$	$(p \to q) \land \neg p$	$\neg q$	$((p \to q) \land \neg p) \to \neg q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Example of the Fallacy of Denying the Hypothesis

If it rains, then it is cloudy.
It is not the case that it rains.
Therefore, it is not cloudy (wrong).

r is the proposition "it rains."
c is the proposition "it is cloudy."

$$\begin{array}{rl} & r \rightarrow c \\ & \neg r \\ \hline \therefore & \neg c \text{ (wrong)} \end{array}$$

Outline

1. Rules of Inference
   - Motivation
   - Definitions
   - Rules of Inference
   - Fallacies
   - Using Rules of Inference to Build Arguments
   - Rules of Inference and Quantifiers

Example: Existence of Superman

If Superman were able and willing to prevent evil, then he would so. If Superman were unable to prevent evil, then he would be impotent; if he were unwilling to prevent evil, then he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

- w is "Superman is willing to prevent evil"
- a is "Superman is able to prevent evil"
- i is "Superman is impotent"
- m is "Superman is malevolent"
- p is "Superman prevents evil"
- x is "Superman exists"

Example: Existence of Superman

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- $h1.\ (a \wedge w) \rightarrow p$
- $h2.\ \neg a \rightarrow i$
- $h3.\ \neg w \rightarrow m$
- $h4.\ \neg p$
- $h5.\ x \rightarrow \neg i$
- $h6.\ x \rightarrow \neg m$

Example: Existence of Superman

Argument:

1.	$\neg i \longrightarrow a$	contrapositive of h_2.
2.	$x \longrightarrow a$	h_5 and step 1 with hyp. syll.
3.	$\neg m \longrightarrow w$	contrapositive of h_3.
4.	$x \longrightarrow w$	h_6 ans step 3 with hyp. syll.
5.	$x \longrightarrow (a \wedge w)$	Step 2 and 4 with conjunction.
6.	$x \longrightarrow p$	Step 5 and h_1 with hyp. syll.
7.	$\neg x$	Step 6 and h_4 with *modus tollens*.

Q.E.D.

Outline

Rules of Inference and Quantifiers

There are four rules of inference for quantifiers:

- Universal instantiation (UI),
- Universal generalization (UG),
- Existential instantiation (EI),
- Existential generalization (EG).

Universal Instantiation

$$\frac{\forall x\, P(x)}{\therefore\ P(c)}$$

If a propositional function is true for all element x of the universe of discourse, then it is true for a particular element c of the universe of discourse.

Universal Instantiation and *modus ponens*

The universal instantiation and the *modus ponens* are used together to form the **universal modus ponens**. Example: *All humans have two legs. John Smith is a human. Therefore, John Smith has two legs.*

- $H(x)$ is "x is a human."
- $L(x)$ is "x has two legs."
- j is John Smith, a element of the universe of discourse.

1.	$\forall x\,(H(x) \rightarrow L(x))$	Premise.
2.	$H(j) \rightarrow L(j)$	Universal instantiation from 1.
3.	$H(j)$	Premise.
$\therefore$	$L(j)$	*Modus ponens* from 2. et 3.

Universal Generalization

$$\frac{P(c) \quad \text{for an arbitrary } c}{\therefore\ \forall x\, P(x)}$$

We must first define the universe of discourse. Then, we must show that $P(c)$ is true for an arbitrary, and not a specific, element c of the universe of discourse. We have no control over c and we can not make any other assumptions about c other than it comes from the domain of discourse. The error of adding unwarranted assumptions about the arbitrary element c is common and is an incorrect reasoning.

Existential Instantiation

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \quad \text{for some element } c}$$

The existential instantiation is the rule that allow us to conclude that there is an element c in the universe of discourse for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We can not select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true.

Existential Generalization

$$\frac{P(c) \quad \text{for some element } c}{\therefore\ \exists x\, P(x)}$$

If we know one element c in the universe of discourse for which $P(c)$ is true, therefore we know that $\exists x\, P(x)$ is true.