Numerical Integration

First part: General Formula for Numerical Integration (Trapezoidal)
Learning Results

- How to find out General Formula for the Numerical Integration?
- How to get Trapezoidal Rules from General Formula?
- How to solve the Trapezoidal problem?
Basic Concept (Application of Integration)

Area?

4

6

y = f(x)

x = a

x = b
Basic Concept (Application of Integration)

\[ \int_{x=a}^{x=b} f(x) \, dx = [F(x)]_a^b = F(b) - F(a) \]
\[ y = x^2 \]

\[
\text{Area} = \int_{x=a}^{x=b} f(x) \, dx
\]

\[ = \int_{1}^{2} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{1}^{2}
\]

\[ = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}
\]
\[ y = f(x) \quad h = \frac{b-a}{n} \]

\[ = \frac{upper \ limit - lower \ limit}{n} \]

\[ = \int_{1}^{2} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{1}^{2} \]

\[ = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \]
y = x^2

h = \frac{b-a}{n}

h = \frac{2-1}{6}

h = \frac{1}{6}

x_5 = x_4 + h

x_3 = x_2 + h

x_4 = x_3 + h

x_1 = 1 + h

x_2 = x_1 + h

x_3 = x_2 + h

x_6 = x_5 + h

= \frac{11}{6} + \frac{1}{6} = \frac{12}{6} = 2

= \frac{7}{6} + \frac{1}{6} = \frac{8}{6}

= \frac{9}{6} + \frac{1}{6} = \frac{10}{6}
Let,
\[ I = \int_a^b y \, dx = \int_{x_0}^{x_0 + nh} y \, dx \quad (1) \]

From Newton’s Forward Interpolation Formula, we have,
General formula for Numerical Integration

\[ y(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots + \frac{u(u-1)(u-2) \cdots (u-n+1)}{n!} \Delta^n y_0 \]

where, \( u = \frac{x - x_0}{h} \)
From (1) we get,

\[ I = \int_{x_0}^{x_0+nh} \left[ y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots \right] dx \tag{2} \]

Now we know,

\[ u = \frac{x - x_0}{h} \]

\[ \Rightarrow x = x_0 + uh \quad \therefore \, dx = hdu \]
Limit Change:

When $x = x_0$ then $u = 0$

When $x = x_n$ then $u = n$

Therefore, above equation (2) takes the form,

$$I = \int_0^n \left[ y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots + \text{upto (n+1) terms} \right] hdu$$

$$= \int_0^n \left[ y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots + \text{upto (n+1) terms} \right] du$$
\[ = \int_0^n \left[ y_0 + u \Delta y_0 + \frac{(u^2 - u)}{2!} \Delta^2 y_0 + \frac{(u^2 - u)(u - 2)}{3!} \Delta^3 y_0 + \ldots + \text{upto (n+1)terms} \right] du \]

\[ = \int_0^n \left[ y_0 + u \Delta y_0 + \frac{(u^2 - u)}{2!} \Delta^2 y_0 + \frac{(u^3 - 3u^2 + 2u)}{3!} \Delta^3 y_0 + \ldots + \text{upto (n+1)terms} \right] du \]

\[ = \left[ {y_0 u + \frac{u^2}{2} \Delta y_0 + \frac{1}{2!} \left( \frac{u^3}{3} - \frac{u^2}{2} \right) \Delta^2 y_0 + \frac{1}{3!} \left( \frac{u^4}{4} - u^3 + u^2 \right) \Delta^3 y_0 + \ldots + \text{upto (n+1)terms} } \right]_0^n \]

\[ \therefore I = \int_a^b y dx = \int_{x_0}^{x_0 + nh} y dx = h \left( n y_0 + \frac{n^2}{2} \Delta y_0 + \left( \frac{n^3}{3} - \frac{n^2}{2} \right) \Delta^2 y_0 + \left( \frac{n^4}{4} - n^3 + n^2 \right) \Delta^3 y_0 + \ldots + \text{upto (n+1)terms} \right) \]

This Formula is known as general quadrature formula or General formula for numerical integration and also known as General Gauss-Legendre integration formula for equidistant ordinates.
Note:

1. This formula is used to compute $\int_{a}^{b} f(x) \, dx$

2. Putting $n = 1$ in above equation we obtain Trapezoidal rule

3. Putting $n = 2$ in above equation we obtain Simpson’s $\frac{1}{3}$ Rule

4. Putting $n = 3$ in above equation we obtain Simpson’s $\frac{3}{8}$ Rule
5. Putting $n = 4$ in above equation we obtain Boole’s Rule

6. Putting $n = 6$ in above equation we obtain Weddle’s Rule
The general integration formula is

\[
I = \int_{a}^{b} y \, dx = \int_{x_0}^{x_0 + nh} y \, dx
\]

\[
= h \left( n y_0 + \frac{n^2}{2} \Delta y_0 + \left( \frac{n^3}{3} - \frac{n^2}{2} \right) \frac{\Delta^2 y_0}{2!} + \left( \frac{n^4}{4} - n^3 + n^2 \right) \frac{\Delta^3 y_0}{3!} + \cdots \right) + \text{upto } (n+1) \text{terms}
\]

Setting \( n = 1 \) in above equation and neglecting the second and higher order, we get
\[
\int_{x_0}^{x_0+h} y \, dx = h \left( y_0 + \frac{1}{2} \Delta y_0 \right)
= h \left( y_0 + \frac{1}{2} (y_1 - y_0) \right)
= h \left( y_0 + \frac{1}{2} y_1 - \frac{1}{2} y_0 \right)
= h \left( \frac{1}{2} y_0 + \frac{1}{2} y_1 \right)
\]
Similarly, we can get,

\[
\int_{x_0 + h}^{x_0 + 2h} y \, dx = \frac{h}{2} (y_1 + y_2)
\]

\[
\int_{x_0 + 2h}^{x_0 + 3h} y \, dx = \frac{h}{2} (y_2 + y_3)
\]
\[ \int_{x_0}^{x_0+nh} y \, dx = \frac{h}{2} \left( y_{n-1} + y_n \right) \]

Adding these \( n \) integrals, we get

\[ \int_{x_0}^{x_0+nh} y \, dx = \frac{h}{2} \left( y_0 + y_1 \right) + \frac{h}{2} \left( y_1 + y_2 \right) + \frac{h}{2} \left( y_2 + y_3 \right) + \cdots + \frac{h}{2} \left( y_{n-1} + y_n \right) \]
The above formula is known as the trapezoidal rule for numerical integration.

Shortly we can write,

\[ \int_{x_0}^{x_0+nh} y \, dx = \frac{h}{2} \left[ (y_0 + y_n) + 2 \sum_{k=1}^{n-1} y_k \right] \]
Problem

Evaluate $\int_{0}^{6} f(x) \, dx$ by using trapezoidal rule where the values of $f(x)$ are given by the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y=f(x)</td>
<td>0.146</td>
<td>0.161</td>
<td>0.176</td>
<td>0.190</td>
<td>0.204</td>
<td>0.217</td>
<td>0.230</td>
</tr>
</tbody>
</table>

Solution:
Here upper limit is $b=6$, lower limit is $a=0$ and No. of subintervals $n=6$. 
Now,
\[ h = \frac{6 - 0}{6} = 1 \]

The values of the function \( y \) at each subinterval are given in the tabular form:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x_0 = 0 )</th>
<th>( x_1 = 1 )</th>
<th>( x_2 = 2 )</th>
<th>( x_3 = 3 )</th>
<th>( x_4 = 4 )</th>
<th>( x_5 = 5 )</th>
<th>( x_6 = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = f(x) )</td>
<td>( y_0 = 0.146 )</td>
<td>( y_1 = 0.161 )</td>
<td>( y_2 = 0.176 )</td>
<td>( y_3 = 0.190 )</td>
<td>( y_4 = 0.204 )</td>
<td>( y_5 = 0.217 )</td>
<td>( y_6 = 0.230 )</td>
</tr>
</tbody>
</table>

From trapezoidal rule we have

\[
\int_{a}^{b} f(x) \, dx = \frac{1}{2} \left[ (y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5) \right]
\]

\[
\int_{a}^{b} f(x) \, dx = \frac{1}{2} \left[ (0.146 + 0.230) + 2(0.161 + 0.176 + 0.190 + 0.204 + 0.217) \right]
\]
\[ \int_{0}^{6} f(x) \, dx = 1.136 \]
Calculate the value of the integral \( I = \int_{0}^{1} \frac{x \, dx}{1 + x^2} \) by taking seven equidistant ordinates, using the trapezoidal rule. Find the exact value of \( I \) and then compare and comment on it.