Numerical Integration

2nd part: Simpson’s one third

Preparing by Protima Dash
Simpson’s $\frac{1}{3}$ Rule

The general Integration Formula,

$$ I = \int_{a}^{b} y \, dx = \int_{x_0}^{x_0+nh} y \, dx $$
\[ I = h \left[ n y_0 + \frac{n^2}{2} \Delta y_0 + \left( \frac{n^3}{3} - \frac{n^2}{2} \right) \frac{\Delta^2 y_0}{2!} + \left( \frac{n^4}{4} - n^3 + n^2 \right) \frac{\Delta^3 y_0}{3!} + \ldots \ldots \ldots + \text{upto} \ (n + 1) \ \text{terms} \right] \]

Setting \( n = 2 \) in above equation and neglecting the third and higher order, we get
Simpson’s $\frac{1}{3}$ Rule

\[\int_{x_0}^{x_0+2h} y \, dx = h \left( 2y_0 + \frac{2^2}{2} \Delta y_0 + \left( \frac{2^3}{3} - \frac{2^2}{2} \right) \frac{\Delta^2 y_0}{2!} \right)\]

\[= h \left( 2y_0 + 2\Delta y_0 + \left( \frac{8}{3} - 2 \right) \frac{\Delta^2 y_0}{2} \right)\]

\[= h \left( 2y_0 + 2\Delta y_0 + \frac{\Delta^2 y_0}{3} \right)\]
Simpson’s \( \frac{1}{3} \) Rule

\[
= h \left( 2y_0 + 2(y_1 - y_0) + \frac{1}{3} (\Delta y_1 - \Delta y_0) \right)
\]

\[
= h \left( 2y_0 + 2(y_1 - y_0) + \frac{1}{3} \{(y_2 - y_1) - (y_1 - y_0)\} \right)
\]

\[
= h \left( 2y_0 + 2y_1 - 2y_0 + \frac{1}{3} (y_2 - 2y_1 + y_0) \right)
\]
Simpson’s $\frac{1}{2}$ Rule

\[
\frac{h}{3}(6y_0 + 6y_1 - 6y_0 + (y_2 - 2y_1 + y_0))
\]

\[
\int_{x_0}^{x_0+2h} y \, dx = \frac{h}{3}(y_0 + 4y_1 + y_2)
\]

Similarly, we can write,
Simpson’s $\frac{1}{3}$ Rule

\[ \int_{x_0+2h}^{x_0+4h} y \, dx = \frac{h}{3} (y_2 + 4y_3 + y_4) \]

\[ \int_{x_0+4h}^{x_0+6h} y \, dx = \frac{h}{3} (y_4 + 4y_5 + y_6) \]
Simpson’s $\frac{1}{3}$ Rule

\[ \int_{x_0+(n-2)h}^{x_0+nh} y \, dx = \frac{h}{3} \left( y_{n-2} + 4y_{n-1} + y_n \right) \]

Now adding the $n$ integrals, we can write
Simpson’s $\frac{1}{3}$ Rule

\[ \int_{x_0}^{x_0 + nh} y \, dx = \frac{h}{3} \left( y_0 + 4y_1 + y_2 \right) + \frac{h}{3} \left( y_2 + 4y_3 + y_4 \right) + \frac{h}{3} \left( y_4 + 4y_5 + y_6 \right) + \cdots + \frac{h}{3} \left( y_{n-2} + 4y_{n-1} + y_n \right) \]

\[ = \frac{h}{3} \left[ y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \cdots + y_{n-2} + 4y_{n-1} + y_n \right] \]
Simpson’s $\frac{1}{3}$ Rule

\[
\int_{x_0}^{x_0 + nh} y \, dx = \frac{h}{3} \left[ (y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2}) \right]
\]
Simpson’s $\frac{1}{3}$ Rule

The above formula is known as the Simpson’s $1/3$ rule for numerical integration.

Shortly we can write,

$$\int_{x_0}^{x_0+nh} y \, dx = \frac{h}{3} \left[ (y_0 + y_n) + 4 \sum_{k=1,3,5,\ldots}^{n-1} y_k + 2 \sum_{k=2,4,6,\ldots}^{n-2} y_k \right]$$

Note:
This formula is used only when the number of partitions of the interval of integration is even.
Mathematical Problems

**Problem 01:** Compute \( \int_{1}^{2} x^2 \, dx \) by Simpson’s one third rule and compare with exact value.

**Solution:**

Given that the function is, \( \int_{1}^{2} x^2 \, dx \)

Here upper limit is \( b = 2 \), lower limit is \( a = 1 \)

and number of subintervals \( n = 4 \)
let \( y = f(x) = x^2 \)

Now,
\[
    h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25
\]

The values of the function \( y \) at each subinterval are given in the tabular form:
From the **Simpson’s \( \frac{1}{3} \) Rule**, we have
\[
\int_{x_0}^{x_0 + nh} y \, dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})]
\]
Now for \( n = 4 \) the above formula reduces to the following form,

\[
\int_1^2 x^2 \, dx = \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]
\]

\[
= \frac{0.25}{3} [(1 + 4) + 4(1.5625 + 3.0625) + 2 \times 2.25]
\]

\[
= \frac{1}{3} [(1 + 4) + 4(1.5625 + 3.0625) + 2 \times 2.25]
\]

\[
= \frac{1}{3} [5 + 4(4.625) + 4.5]
\]

\[
= \frac{1}{3} [5 + 18.5 + 4.5]
\]

\[
= \frac{1}{3} [28]
\]

\[
= 9.3333
\]

\[
y_0 = 1
\]

\[
y_4 = 4
\]

\[
y_1 = 1.5625
\]

\[
y_2 = 2.25
\]

\[
y_3 = 3.0625
\]
Mathematical Problems

\[ \int_{1}^{2} x^2 \, dx = \frac{7}{3} \]

Now exact value is
\[ \int_{1}^{2} x^2 \, dx = \left[ \frac{x^3}{3} \right]_{1}^{2} = \frac{1}{3} \left( 2^3 - 1^3 \right) = \frac{7}{3} \]

It is shown that exact result and Simpson’s \( \frac{1}{3} \) Rule’s result are exactly same so there is no error between two results.
Problem 02:

Compute the definite integral $\int_{0.2}^{1.4} \left( \sin x - \ln x + e^x \right) \, dx$

by using various rules using 6 equidistant sub-intervals correct up to three decimal places.
Solution:

Given that the function is, \[ \int_{0.2}^{1.4} \left( \sin x - \ln x + e^x \right) dx \]

Here upper limit is \( b = 1.4 \), lower limit is \( a = 0.2 \)

No. of subintervals \( n = 6 \)
Mathematical Problems

Let,

\[ y = f(x) = \sin x - \ln x + e^x. \]

Now,

\[ h = \frac{1.4 - 0.2}{6} = 0.2 \]

The values of the function \( y \) at each subinterval are given in the tabular form:
### Mathematical Problems

<table>
<thead>
<tr>
<th>x</th>
<th>( x_0 = 0.2 )</th>
<th>( x_1 = 0.4 )</th>
<th>( x_2 = 0.6 )</th>
<th>( x_3 = 0.8 )</th>
<th>( x_4 = 1.0 )</th>
<th>( x_5 = 1.2 )</th>
<th>( x_6 = 1.4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>( y_0 = 3.0295 )</td>
<td>( y_1 = 2.7975 )</td>
<td>( y_2 = 2.8975 )</td>
<td>( y_3 = 3.1660 )</td>
<td>( y_4 = 3.5597 )</td>
<td>( y_5 = 4.0698 )</td>
<td>( y_6 = 4.7041 )</td>
</tr>
</tbody>
</table>

From the **Simpson’s \( \frac{1}{3} \) Rule**, we have
\[
\int_{x_0}^{x_0 + nh} y \, dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})]
\]
Now for $n = 6$ the above formula reduces to the following form, 

$$\int_{0.2}^{1.4} \left( \sin x + \ln x - e^x \right) dx$$

$$= \frac{h}{3} \left[ (y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$$= \frac{0.2}{3} \left[ (3.0295 + 4.7041) + 4(2.7975 + 3.1660 + 4.0698) + 2(2.8975 + 3.5597) \right]$$

$$= \frac{0.2}{3} \left[ 7.7336 + 40.1332 + 12.9144 \right]:$$
\[
\int_{0.2}^{1.4} \left( \sin x + \ln x - e^x \right) dx = 4.05208
\]
1. Calculate the value of the integral \( I = \int_{0}^{1} \frac{x \, dx}{1 + x^2} \) by taking seven equidistant ordinates, using the Simpson’s 1/3 rule and trapezoidal rule. Find the exact value of I and then compare and comment on it.
Thank You