Lecture # 4 &5

Differential Equations:

First order and first degree D.E.

(Variable Separation)

First order first degree D.E.

A differential equation of the form $M(x,y)dy + N(x,y)dx = 0$

or $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ is called a first order and first degree

differential equation. For example -

$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ is a first order and first degree D.E

To solve this kind of differential equation, first we have to separate the variables and after integrating we will find its solution.

First order first degree D.E.

Problem :

Solve the differential equations

$(a)\ \dfrac{dy}{dx}=\dfrac{1+y^2}{1+x^2}$ $\quad (b)\ \sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy$

$(c)\ y-x\dfrac{dy}{dx}=a\left(y^2+\dfrac{dy}{dx}\right)$ $\quad (d)\ (e^y+1)\cos x\,dx+e^y\sin x\,dy=0$

Solution: (a)

Given that

$$\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$$

$$\Rightarrow (1+x^2)dy=(1+y^2)dx$$

Separating the variables, we get

$$\frac{dy}{1+y^2}=\frac{dx}{1+x^2}$$

First order first degree D.E.

Integrating both sides, we get

$$\int \frac{1}{1+y^2}dy = \int \frac{1}{1+x^2}dx$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + c$$

$$\Rightarrow \tan^{-1} y - \tan^{-1} x = \tan^{-1} A$$

$$\left[\tan^{-1} A = c = constant\right]$$

$$\Rightarrow \tan^{-1} \frac{y-x}{1+xy} = \tan^{-1} A$$

$$\Rightarrow \frac{y-x}{1+xy} = A$$

$$\Rightarrow y - x = A(1+xy) \qquad \text{(Ans.)}$$

First order first degree D.E.

Solution (b):

Given that

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

$$\Rightarrow \sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$$

Separating the variables, we get

$$\frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

Integrating both sides, we get

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy$$

$$\log \tan x = -\log \tan y + c$$

$$[\because \int \frac{f'(x)}{f(x)} dx = \log f(x)]$$

First order first degree D.E.

$$\Rightarrow \log \tan x + \log \tan y = \log A$$

$$\Rightarrow \log(\tan x \tan y) = \log A$$

$$\Rightarrow \tan x \tan y = A \quad (Ans)$$

First order first degree D.E.

Solution (c):

Given that

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$$

$$\Rightarrow y - ay^2 = x\frac{dy}{dx} + a\frac{dy}{dx}$$

$$\Rightarrow y(1 - ay) = (x + a)\frac{dy}{dx}$$

First order first degree D.E.

$$\Rightarrow y(1-ay)dx=(x+a)dy$$

Separating the variables, we get

$$\frac{dx}{x+a}=\frac{dy}{y(1-ay)}$$

$$\Rightarrow \frac{dx}{x+a}=\left(\frac{1}{y}+\frac{a}{1-ay}\right)dy$$

$$\Rightarrow \frac{dx}{x+a}=\left(\frac{1}{y}-\frac{-a}{1-ay}\right)dy$$

Integrating both sides, we get

$$\log(x+a)+c=\log y-\log(1-ay)$$

$$[\because \int \frac{f^{'}(x)}{f(x)}dx=\log f(x)]$$

First order first degree D.E.

$$\log(x+a)+\log A=\log y-\log(1+ay)$$

$$\Rightarrow \log A(x+a)=\log\frac{y}{1+ay}$$

$$\Rightarrow A(x+a)=\frac{y}{1+ay}\quad \text{(Ans.)}$$

First order first degree D.E.

Solution (d):

Given that

$$(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$$

$$\Rightarrow (e^y+1)\cos x\,dx = -e^y \sin x\,dy$$

Separating the variables, we get

$$\frac{\cos x}{\sin x}dx = -\frac{e^y}{1+e^y}dy$$

Integrating both sides, we get

$$\log\sin x = -\log(1+e^y)+c$$

$$[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)]$$

First order first degree D.E.

$$\Rightarrow \log \sin x = -\log(1+e^{y}) + \log A$$

$$\Rightarrow \log \sin x = \log \frac{A}{1+e^{y}}$$

$$\Rightarrow \sin x = \frac{A}{1+e^{y}} \qquad \text{(Ans.)}$$

Home Task: From Page 7 (attached .pdf in BLC)

First order first degree D.E.

Art.: Equations Reducible to the form in which variables are separable.

If an D.E. equation is of the form $\frac{dy}{dx} = f(ax+by+c)$ then

we let, $ax+by+c=v$

$$\therefore\ a+b\frac{dy}{dx}=\frac{dv}{dx}\quad i.e.\ \frac{dy}{dx}=\frac{1}{b}\left(\frac{dv}{dx}-a\right)$$ to separate the variables

Solve the following D.Es:

(a) $\frac{dy}{dx}=(4x+y+1)^2$ (b) $(x-y)^2\frac{dy}{dx}=a^2$

(c) $(x+y)^2\frac{dy}{dx}=a^2$ (d) $\frac{dy}{dx}=\sin(x+y)+\cos(x+y)$

First order first degree D.E.

Solution (a): Given that

$$\frac{dy}{dx} = (4x + y + 1)^2 \text{..........}(1)$$

Let $4x + y + 1 = v$ then

$$4 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dv}{dx} - 4$$

Now from (1), we get

$$\frac{dv}{dx} - 4 = v^2$$

$$\Rightarrow \frac{dv}{dx} = v^2 + 4$$

First order first degree D.E.

$$dv = (v^2 + 4)dx$$

Now separating the variables, we get

$$\Rightarrow \frac{dv}{v^2 + 4} = dx$$

Now integrating both sides, we get

$$\int \frac{1}{v^2 + 2^2} dv = \int dx$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{v}{2} = x + c$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{4x + y + 1}{2} = x + c \qquad \text{(Ans.)}$$

First order first degree D.E.

Solution (b): Given that

$$(x-y)^2\frac{dy}{dx}=a^2 \;..........(1)$$

Let $x-y=v$

Then, $1-\frac{dy}{dx}=\frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx}=1-\frac{dv}{dx}$$

Now from (1), we get

$$v^2\left(1-\frac{dv}{dx}\right)=a^2$$

First order first degree D.E.

$$\Rightarrow 1-\frac{dv}{dx}=\frac{a^2}{v^2}$$

$$\Rightarrow \frac{dv}{dx}=1-\frac{a^2}{v^2}$$

$$\Rightarrow \frac{dv}{dx}=\frac{v^2-a^2}{v^2}$$

$$\Rightarrow v^2dv=(v^2-a^2)dx$$

Now, separating the variables, we get

$$\frac{v^2}{v^2-a^2}dv=dx$$

$$\Rightarrow \frac{(v^2-a^2)+a^2}{v^2-a^2}dv=dx$$

$$\Rightarrow \left(1+\frac{a^2}{v^2-a^2}\right)dv=dx$$

First order first degree D.E.

$$\Rightarrow \left(1+\frac{a^2}{v^2-a^2}\right)dv = dx$$

Integrating both sides, we get

$$\int\left(1+\frac{a^2}{v^2-a^2}\right)dv = \int dx$$

$$\Rightarrow v + a^2\frac{1}{2a}\log\frac{v-a}{v+a} = x + c$$

$$\left[\because \int\frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}\right]$$

$$\Rightarrow x - y + \frac{a}{2}\log\frac{x-y-a}{x-y+a} = x + c$$

$$\Rightarrow -y + \frac{a}{2}\log\frac{x-y-a}{x-y+a} = c$$

$$\Rightarrow y - \frac{a}{2}\log\frac{x-y-a}{x-y+a} = A$$