Homogeneous Differential Equation

An equation of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ in which $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of x and y of the same degree can be reduced to an equation in which variables are separable by putting $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Solve: $\frac{dy}{dx} = \frac{y}{x} + \cos\frac{y}{x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{y}{x} + \cos\frac{y}{x}$..............(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = v + \cos v$

$or, x\frac{dv}{dx} = \cos v$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{dv}{\cos v}$

$or, \frac{dx}{x} = \sec v dv$

Now integrating,

$\int \frac{dx}{x} = \int \sec v dv$

$or, \ln x = \ln|\sec v + \tan v| + \ln c$

$or, \ln x = \ln\{c|\sec v + \tan v|\}$

$or, x = c|\sec v + \tan v|$

$\therefore x = c\left|sec\frac{y}{x} + \tan\frac{y}{x}\right|$

Solve: $\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$.............(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = v + \sec v$

$or, x\frac{dv}{dx} = \sec v$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{dv}{\sec v}$

$or, \frac{dx}{x} = \cos v dv$

Now integrating,

$\int \frac{dx}{x} = \int \cos v dv$

$or, \ln x = \sin v + \ln c$

$or, \ln x = \ln e^{\sin v} + \ln c$

$or, \ln x = \ln\left(c. e^{\sin v}\right)$

$or, x = c. e^{\sin v}$

$\therefore x = c. e^{sin\left(\frac{y}{x}\right)}$

Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ …………(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = v + \tan v$

$or, x\frac{dv}{dx} = \tan v$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{dv}{\tan v}$

$or, \frac{dx}{x} = \cot v dv$

Now integrating,

$\int \frac{dx}{x} = \int \cot v dv$

$or, \ln x = \ln(\sin v) + \ln c$

$or, \ln x = \ln(c. \sin v)$

$or, x = c. \sin v$

$\therefore\ x = c. \sin\left(\frac{y}{x}\right)$

Solve: $\frac{dy}{dx} = \frac{y}{x} + \cot\frac{y}{x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{y}{x} + \cot\frac{y}{x}$……..(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = v + \cot v$

$or, x\frac{dv}{dx} = \cot v$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{dv}{\cot v}$

$or, \frac{dx}{x} = \tan v$

Now integrating,

$\int \frac{dx}{x} = \int \tan v dv$

$or, \ln x = -\ln|\cos v| + \ln c$

$or, \ln x = \ln \frac{c}{|\cos v|}$

$or, x = \frac{c}{|\cos v|}$

$\therefore\ x = \frac{c}{\left|\cos\left(\frac{y}{x}\right)\right|}$

Solve: $\frac{dy}{dx} = \frac{y}{x} + cosec\ \frac{y}{x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{y}{x} + cosec\ \frac{y}{x}$……….(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = v + \operatorname{cosec} v$

$or, x\frac{dv}{dx} = cosec\ v$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{dv}{cosec\ v}$

$or, \frac{dx}{x} = \sin v dv$

Now integrating,

$\int \frac{dx}{x} = \int sinvdv$

$or, \ln x = -\cos v + \ln c$

$or, \ln x = \ln e^{-\cos v} + \ln c$

$or, \ln x = \ln(c.e^{-\cos v})$

$or, x = c.e^{-\cos v}$

$\therefore\ x = c.e^{-\cos\left(\frac{y}{x}\right)}$

Solve: $\frac{dy}{dx} = \frac{x+y}{x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{x+y}{x}$..............(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = \frac{x+vx}{x}$

$or, v + x\frac{dv}{dx} = \frac{x(1+v)}{x}$

$or, v + x\frac{dv}{dx} = 1 + v$

$or, x\frac{dv}{dx} = 1$

Separating the variables we obtain,

$\frac{dx}{x} = dv$

Now integrating,

$\int \frac{dx}{x} = \int dv$

$or, \ln x = v + \ln c$

$or, \ln x = \ln e^{v} + \ln c$

$or, \ln x = \ln(c. e^{v})$

$or, x = c. e^{v}$

$\therefore x = c. e^{\frac{y}{x}}$

Solve: $\frac{dy}{dx} = \frac{2x+y}{2x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{2x+y}{2x}$..............(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = \frac{2x+vx}{2x}$

$or, v + x\frac{dv}{dx} = \frac{x(2+v)}{2x}$

$or, v + x\frac{dv}{dx} = \frac{2+v}{2}$

$or, x\frac{dv}{dx} = \frac{2+v}{2} - v$

$or, x\frac{dv}{dx} = \frac{2-v}{2}$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{2}{2-v}dv$

Now integrating,

$\int \frac{dx}{x} = 2\int \frac{dv}{2-v}$

$or, \ln x = -2\ln(2-v) + \ln c$

$or, \ln x + 2\ln(2-v) = \ln c$

$or, \ln x + \ln(2-v)^2 = \ln c$

$or, \ln\{x.(2-v)^2\} = \ln c$

$or, x.\left(2-\frac{y}{x}\right)^2 = c$

Solve: $\frac{dy}{dx} = \frac{x+2y}{2x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{x+2y}{2x}$…………(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = \frac{x+2vx}{2x}$

$or, v + x\frac{dv}{dx} = \frac{x(1+2v)}{2x}$

$or, x\frac{dv}{dx} = \frac{1+2v}{2} - v$

$or, x\frac{dv}{dx} = \frac{1}{2}$

Separating the variables we obtain,

$\frac{dx}{x} = 2dv$

Now integrating,

$\int \frac{dx}{x} = 2\int dv$

$or, \ln x = 2v + \ln c$

$or, \ln x = \ln e^{2v} + \ln c$

$or, \ln x = \ln(ce^{2v})$

$or, x = ce^{2v}$

$\therefore x = ce^{2\frac{y}{x}}$

Solve: $\frac{dy}{dx} = \frac{x+y}{2x}$

Solution: Given that,

$\frac{dy}{dx} = \frac{x+y}{2x}$............(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = \frac{x+vx}{2x}$

$or, v + x\frac{dv}{dx} = \frac{x(1+v)}{2x}$

$or, x\frac{dv}{dx} = \frac{1+v}{2} - v$

$or, x\frac{dv}{dx} = \frac{1-v}{2}$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{2}{1-v} dv$

Now integrating,

$\int \frac{dx}{x} = \int \frac{2}{1-v} dv$

$or, \ln x = -2\ln(1-v) + \ln c$

$or, \ln x + 2\ln(1-v) = \ln c$

$or, \ln\{x(1-v)^2\} = \ln c$

$or, x(1-v)^2 = c$

$\therefore x\left(1-\frac{y}{x}\right)^2 = c$

Solve: $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

Solution: Given that,

$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$..............(i)

Let $y = vx$

$or, \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) we get,

$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x.vx}$

$or, v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2vx^2}$

$or, x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$

$or, x\frac{dv}{dx} = \frac{1-v^2}{2v}$

Separating the variables we obtain,

$\frac{dx}{x} = \frac{2v}{1-v^2} dv$

Now integrating,

$\int \frac{dx}{x} = \int \frac{2v}{1-v^2} dv$

$or, \ln x = -\ln(1-v^2) + \ln c$

$$or, \ln x + \ln(1 - v^2) = \ln c$$

$$or, \ln\{x(1 - v^2)\} = \ln c$$

$$or, x(1 - v^2) = c$$

$$or, x\left(1 - \frac{y^2}{x^2}\right) = c$$

$$or, 1 - \frac{y^2}{x^2} = \frac{c}{x}$$

$$or, 1 - \frac{c}{x} = \frac{y^2}{x^2}$$

$$or, y^2 = x^2\left(1 - \frac{c}{x}\right)$$

$$\therefore y = \pm x\sqrt{\left(1 - \frac{c}{x}\right)}$$

Problems for Solution

(i) $\frac{dy}{dx} = \frac{3xy}{y^2+3x^2}$

(ii) $\frac{dy}{dx} = \frac{5x-y}{5x}$

(iii) $(x + y)dy = (x - y)dx$

(iv) $\frac{dy}{dx} = \frac{x^4+y^4}{xy^3}$

(v) $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

(vi) $\frac{dy}{dx} = \frac{2xy}{y^2-x^2}$