Linear Differential Equations with Constant Coefficients

Linear Differential Equations: A differential equation of the form,

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots+P_n y=Q \dots\ \dots\ .. \ (1)$$

where, $P_1,\ P_2,\ \dots\ P_n$ and Q are functions of x or, constants, is called a linear differential equation of nth order.

If $P_1,\ P_2,\ \dots\ P_n$ are all constants (not functions of x) and Q is function of x or constant, then the equation is called a linear differential equation with constant coefficients.

If the right-hand term Q (non-homogeneous term) is identically zero, then the equation reduces to,

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots+P_n y=0 \dots\ \dots\ .. \ (2)$$

and it is called a linear homogeneous differential equation.

The general solution of equation (2) will be,

a). $y=c_1e^{m_1x}+c_2e^{m_2x}+\dots\dots+c_ne^{m_nx}$ if the roots $i.e,\ m_1, m_2, \dots\dots m_n$ are real and distinct.

b). $y=\left(c_1+c_2x+\dots\dots+c_nx^{n-1}\right)e^{mx}$ if the roots $i.e,\ m_1, m_2, \dots\dots m_n$ are real and equal.

c). $y=\left(c_1\cos bx+c_2\sin bx\right)e^{ax}$ if the roots are complex $\left(a\pm ib\right)$ and distinct.

d). $y=\left\{\left(c_1+c_2x\right)\cos bx+\left(c_3+c_4x\right)\sin bx\right\}e^{ax}$ if the roots are complex $\left(a\pm ib\right)$ and repeated.

Problem-01: Solve $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$.

OR,

$$D^3y-6D^2y+11Dy-6y=0$$

Solution: Given that,

$$\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0 \ldots \; \ldots \; \ldots \; (1)$$

Let, $y=e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^3e^{mx}-6m^2e^{mx}+11me^{mx}-6e^{mx}=0$$

$$\Rightarrow e^{mx}\left(m^3-6m^2+11m-6\right)=0$$

$$\Rightarrow m^3-6m^2+11m-6=0 \;\; ; \;\; \sin ce \;\; e^{mx}\neq 0$$

$$\Rightarrow m^3-m^2-5m^2+5m+6m-6=0$$

$$\Rightarrow m^2\left(m-1\right)-5m\left(m-1\right)+6\left(m-1\right)=0$$

$$\Rightarrow\left(m-1\right)\left(m^2-5m+6\right)=0$$

$$\Rightarrow\left(m-1\right)\left(m^2-3m-2m+6\right)=0$$

$$\Rightarrow\left(m-1\right)\left\{m\left(m-3\right)-2\left(m-3\right)\right\}=0$$

$$\Rightarrow\left(m-1\right)\left(m-2\right)\left(m-3\right)=0$$

$$\therefore \;\; m-1=0 \;\; ; \;\; m-2=0 \;\; ; \;\; m-3=0$$

$$\Rightarrow \;\; m=1 \;\; ; \;\; m=2 \;\; ; \;\; m=3$$

The general solution is,

$$y=c_1e^{x}+c_2e^{2x}+c_3e^{3x}$$

where, c_1, c_2, c_3 are arbitrary constants.

Problem-02: Solve $\dfrac{d^3y}{dx^3}-13\dfrac{dy}{dx}-12y=0$

OR,

$$D^3y-13Dy-12y=0$$

Solution: Given that,

$$\frac{d^3y}{dx^3}-13\frac{dy}{dx}-12y=0 \ldots \; \ldots \; \ldots \; (1)$$

Let, $y=e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^3e^{mx} - 13me^{mx} - 12e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^3 - 13m - 12\right) = 0$$

$$\Rightarrow m^3 - 13m - 12 = 0 \;\; ; \;\; \sin ce\; e^{mx} \neq 0$$

$$\Rightarrow m^3 + m^2 - m^2 - m - 12m - 12 = 0$$

$$\Rightarrow m^2\left(m+1\right) - m\left(m+1\right) - 12\left(m-1\right) = 0$$

$$\Rightarrow \left(m+1\right)\left(m^2 - m - 12\right) = 0$$

$$\Rightarrow \left(m+1\right)\left(m^2 - 4m + 3m - 12\right) = 0$$

$$\Rightarrow \left(m+1\right)\left\{m\left(m-4\right) + 3\left(m-4\right)\right\} = 0$$

$$\Rightarrow \left(m+1\right)\left(m+3\right)\left(m-4\right) = 0$$

$$\therefore \;\; m+1 = 0 \;\; ; \;\; m+3 = 0 \;\; ; \;\; m-4 = 0$$

$$\Rightarrow \;\; m = -1 \;\; ; \;\; m = -3 \;\; ; \;\; m = 4$$

The general solution is,

$$y = c_1e^{-x} + c_2e^{-3x} + c_3e^{4x}$$

where, c_1, c_2, c_3 are arbitrary constants.

Problem-03: Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$

$$OR,$$

$$D^2y - 4Dy + 4y = 0$$

Solution: Given that,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 \ldots \ldots \ldots (1)$$

Let, $y = e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^2e^{mx} - 4me^{mx} + 4e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 4m + 4\right) = 0$$

$$\Rightarrow m^2 - 4m + 4 = 0 \;\; ; \;\; \sin ce\; e^{mx} \neq 0$$

$$\Rightarrow (m-2)(m-2)=0$$

$$\therefore \quad m-2=0 \;\; ; \;\; m-2=0$$

$$\Rightarrow \;\; m=2 \;\; ; \;\; m=2$$

The general solution is,

$$y=(c_1+c_2x)e^{2x}$$

where, c_1, c_2 are arbitrary constants.

Problem-04: Solve $\frac{d^4y}{dx^4}-\frac{d^3y}{dx^3}-9\frac{d^2y}{dx^2}-11\frac{d\ y}{dx}-4y=0$.

OR

$$D^4y-D^3y-9D^2y-11Dy-4y=0$$

Solution: Given that,

$$\frac{d^4y}{dx^4}-\frac{d^3y}{dx^3}-9\frac{d^2y}{dx^2}-11\frac{d\ y}{dx}-4y=0 \ldots \ldots \ldots (1)$$

Let, $y=e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^4e^{mx}-m^3e^{mx}-9m^2e^{mx}-11me^{mx}-4e^{mx}=0$$

$$\Rightarrow e^{mx}\left(m^4-m^3-9m^2-11m-4\right)=0$$

$$\Rightarrow m^4-m^3-9m^2-11m-4=0 \; ; \;\; \sin ce \; e^{mx}\neq 0$$

$$\Rightarrow m^4-4m^3+3m^3-12m^2+3m^2-12m+m-4=0$$

$$\Rightarrow m^3(m-4)+3m^2(m-4)+3m(m-4)+1(m-4)=0$$

$$\Rightarrow (m-4)\left(m^3+3m^2+3m+1\right)=0$$

$$\Rightarrow (m-4)(m+1)^3=0$$

$$\therefore \quad m-4=0 \;\; ; \;\; m+1=0 \; ; \; m+1=0 \; ; \; m+1=0$$

$$\Rightarrow \;\; m=4 \;\; ; \;\; m=-1 \; ; \; m=-1 \; ; \; m=-1$$

The general solution is,

$$y=\left(c_1+c_2x+c_3x^2\right)e^{-x}+c_4e^{4x}$$

where, c_1, c_2, c_3, c_4 are arbitrary constants.

Problem-05: Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$.

$$OR,$$

$$D^2y-2Dy+2y=0$$

Solution: Given that,

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0 \ldots \ \ldots \ \ldots \ (1)$$

Let, $y=e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^2e^{mx}-2me^{mx}+2e^{mx}=0$$

$$\Rightarrow e^{mx}\left(m^2-2m+2\right)=0$$

$$\Rightarrow m^2-2m+2=0 \quad ; \quad \sin ce \ e^{mx}\neq 0$$

$$\Rightarrow m^2-2m+2=0$$

$$\therefore \quad m=\frac{2\pm\sqrt{4-8}}{2}$$

$$=\frac{2\pm\sqrt{-4}}{2}$$

$$=\frac{2\pm\sqrt{4i^2}}{2}$$

$$=\frac{2\pm 2i}{2}$$

$$=1\pm i$$

The general solution is,

$$y=c_1e^{(1+i)x}+c_2e^{(1-i)x}$$

$$=c_1e^x.e^{ix}+c_2e^x.e^{-ix}$$

$$=e^x\left[e^{ix}+c_2e^{-ix}\right]$$

$$=e^x\left[c_1\left(\cos x+i\sin x\right)+c_2\left(\cos x-i\sin x\right)\right]$$

$$=e^x\left[\left(c_1+c_2\right)\cos x+i\left(c_1-c_2\right)\sin x\right]$$

$$= e^{x}\left[A\cos x + B\sin x\right] \quad ; \quad putting,\ A = \left(c_1 + c_2\right)\ and\ B = i\left(c_1 - c_2\right)$$

where, A, B are arbitrary constants.

Problem-06: Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$.

OR,

$$D^2y - 4Dy + 13y = 0$$

Solution: Given that,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0 \dots \dots \dots (1)$$

Let, $y = e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^2e^{mx} - 4me^{mx} + 13e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 4m + 13\right) = 0$$

$$\Rightarrow m^2 - 4m + 13 = 0 \quad ; \quad \text{since } e^{mx} \neq 0$$

$$\Rightarrow m^2 - 4m + 13 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36i^2}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

The general solution is,

$$y = c_1e^{(2+3i)x} + c_2e^{(2-3i)x}$$

$$= c_1e^{2x}.e^{3ix} + c_2e^{2x}.e^{-3ix}$$

$$= e^{2x}\left[e^{3ix} + c_2e^{-3ix}\right]$$

$$= e^{2x}\left[c_1\left(\cos 3x + i\sin 3x\right) + c_2\left(\cos 3x - i\sin 3x\right)\right]$$

$$= e^{2x}\left[\left(c_1 + c_2\right)\cos 3x + i\left(c_1 - c_2\right)\sin 3x\right]$$

$$= e^{2x}\left[A\cos 3x + B\sin 3x\right] \quad ; \quad putting,\ A = \left(c_1 + c_2\right)\ and\ B = i\left(c_1 - c_2\right)$$

where, A, B are arbitrary constants.

Problem-07: Solve $\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} + 6y = 0$.

OR,

$$D^4y + 5D^2y + 6y = 0$$

Solution: Given that,

$$\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} + 6y = 0 \ \ldots \ \ldots \ \ldots \ (1)$$

Let, $y = e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^4e^{mx} + 5m^2e^{mx} + 6e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^4 + 5m^2 + 6\right) = 0$$

$$\Rightarrow m^4 + 5m^2 + 6 = 0 \quad ; \quad \text{since } e^{mx} \neq 0$$

$$\Rightarrow m^4 + 5m^2 + 6 = 0$$

$$\Rightarrow m^4 + 3m^2 + 2m^2 + 6 = 0$$

$$\Rightarrow m^2\left(m^2 + 3\right) + 2\left(m^2 + 3\right) = 0$$

$$\Rightarrow \left(m^2 + 2\right)\left(m^2 + 3\right) = 0$$

$$\therefore \quad m^2 + 2 = 0 \quad ; \quad m^2 + 3 = 0$$

$$\Rightarrow \quad m^2 = -2 \quad ; \quad m^2 = -3$$

$$\Rightarrow \quad m = \pm\sqrt{2}\,i \quad ; \quad m = \pm\sqrt{3}i$$

The general solution is,

$$y = c_1e^{\sqrt{2}ix} + c_2e^{-\sqrt{2}ix} + c_3e^{\sqrt{3}ix} + c_4e^{-\sqrt{3}ix}$$

$$= c_1\left(\cos\sqrt{2}x + i\sin\sqrt{2}x\right) + c_2\left(\cos\sqrt{2}x - i\sin\sqrt{2}x\right)$$
$$+ c_3\left(\cos\sqrt{3}x + i\sin\sqrt{3}x\right) + c_4\left(\cos\sqrt{3}x - i\sin\sqrt{3}x\right)$$

$$= (c_1 + c_2)\cos\sqrt{2}x + i(c_1 - c_2)\sin\sqrt{2}x + (c_3 + c_4)\cos\sqrt{3}x + i(c_3 - c_4)\sin\sqrt{3}x$$

$$= A\cos\sqrt{2}x + B\sin\sqrt{2}x + C\cos\sqrt{3}x + D\sin\sqrt{3}x$$
$$putting,\ A = (c_1 + c_2)\ ;\ B = i(c_1 - c_2)\ ;\ C = (c_3 + c_4)\ ;\ D = i(c_3 - c_4)$$

where, A, B, C, D are arbitrary constants.

Problem-08: Solve $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{dy}{dx} + y = 0$.

$$OR$$
$$D^4y - D^3y - Dy + y = 0$$

Solution: Given that,

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{dy}{dx} + y = 0 \ldots\ \ldots\ \ldots\ (1)$$

Let, $y = e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^4e^{mx} - m^3e^{mx} - me^{mx} + e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^4 - m^3 - m + 1\right) = 0$$

$$\Rightarrow m^4 - m^3 - m + 1 = 0\ \ ;\ \ \text{since}\ e^{mx} \neq 0$$

$$\Rightarrow m^4 - m^3 - m + 1 = 0$$

$$\Rightarrow m^3(m-1) - 1(m-1) = 0$$

$$\Rightarrow (m-1)\left(m^3 - 1\right) = 0$$

$$\Rightarrow (m-1)(m-1)\left(m^2 + m + 1\right) = 0$$

$$\therefore\ \ m - 1 = 0\ ;\ m - 1 = 0\ ;\ m^2 + m + 1 = 0$$

$$\Rightarrow\ m = 1\ ;\ m = 1\ ;\ m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3i^2}}{2}$$

$$= \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$$

The general solution is,

$$y = (c_1 + c_2 x)e^x + e^{-\frac{x}{2}}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x\right]$$

where, $c_1,\ c_2,\ c_3,\ c_4$ are arbitrary constants.

Problem-09: Solve $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$.

OR

$$D^4y + 2D^2y + y = 0$$

Solution: Given that,

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0 \ldots \ldots \ldots (1)$$

Let, $y = e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^4e^{mx} + m^2e^{mx} + e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^4 + 2m^2 + 1\right) = 0$$

$$\Rightarrow m^4 + 2m^2 + 1 = 0 \ ; \ \sin ce\ e^{mx} \neq 0$$

$$\Rightarrow \left(m^2 + 1\right)^2 = 0$$

$$\therefore \quad m^2 + 1 = 0 \ ; \ m^2 + 1 = 0$$

$$\Rightarrow \ m^2 = -1 \ ; \ m^2 = -1$$

$$\Rightarrow \ m^2 = i^2 \ ; \ m^2 = i^2$$

$$\Rightarrow \ m = \pm i \ ; \ m = \pm i$$

The general solution is,

$$y=(c_1+c_2x)e^{ix}+(c_3+c_4x)e^{-ix}$$

$$=(c_1+c_2x)(\cos x+i\sin x)+(c_3+c_4x)(\cos x-i\sin x)$$

$$=\left[c_1+c_3+(c_2+c_4)x\right]\cos x+i\left[c_1-c_3+(c_2-c_4)x\right]\sin x$$

$$=(A+Bx)\cos x+(C+Dx)\sin x$$

$$putting,\ A=(c_1+c_2)\ ;\ B=(c_3+c_4)\ ;\ C=i(c_1-c_2)\ ;\ D=i(c_3-c_4)$$

where, A, B, C, D are arbitrary constants.

Problem-10: Solve $\frac{d^6y}{dx^6}+3\frac{d^4y}{dx^4}+3\frac{d^2y}{dx^2}+y=0$.

Solution: Given that,

$$\frac{d^6y}{dx^6}+3\frac{d^4y}{dx^4}+3\frac{d^2y}{dx^2}+y=0 \ldots\ \ldots\ \ldots\ (1)$$

Let, $y=e^{mx}$ be the trial solution.

The auxiliary equation of (1) is,

$$m^6e^{mx}+3m^4e^{mx}+3m^2e^{mx}+e^{mx}=0$$

$$\Rightarrow e^{mx}\ (m^6+3m^4+3m^2+1)=0$$

$$\Rightarrow m^6+3m^4+3m^2+1=0\ \ ;\ \ \sin ce\ e^{mx}\neq 0$$

$$\Rightarrow (m^2+1)^3=0$$

$$\therefore\ \ m^2+1=0\ ;\ m^2+1=0\ ;\ m^2+1=0$$

$$\Rightarrow\ m^2=-1\ ;\ m^2=-1\ ;\ m^2=-1$$

$$\Rightarrow\ m^2=i^2\ ;\ m^2=i^2\ ;\ m^2=i^2$$

$$\Rightarrow\ m=\pm i\ ;\ m=\pm i\ ;\ m=\pm i$$

The general solution is,

$$y=(c_1+c_2x+c_3x^2)\cos x+(c_4+c_5x+c_6x^2)\sin x$$

where, c_1, c_2, c_3, c_4, c_5, c_6 are arbitrary constants.

Exercise:

1. Solve $D^2y - 3Dy + 2y = 0$ $Ans: c_1e^x + c_2e^{2x}$

2.Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ $Ans: c_1e^{2x} + c_2e^{-3x}$

3.Solve $D^2y - 4Dy + y = 0$ $Ans: e^{2x}\left(c_1e^{\sqrt{3}x} + c_2e^{-\sqrt{3}x}\right)$

4.Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$ $Ans: (c_1 + c_2x)e^x + c_3e^{-2x}$

5.Solve $D^4y - 4D^2y + 4y = 0$ $Ans\,(c_1 + c_2x)e^{\sqrt{2}x} + (c_3 + c_4x)e^{-\sqrt{2}x}$

6.Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$ $Ans: (A\cos 2x + B\sin 2x)e^x$

7.Solve $\frac{d^3y}{dx^3} + 8y = 0$ $Ans: c_1e^{-2x} + \left\{c_2\cos\left(\sqrt{3}x\right) + c_3\sin\left(\sqrt{3}x\right)\right\}e^x$

8.Solve $D^4y - 81y = 0$ $Ans: c_1e^{3x} + c_2e^{-3x} + c_3\cos 3x + c_4\sin 3x$

Linear Differential Equations with Constant Coefficients but Right Side non-zero.

Consider a differential equation of the form,

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\dots\dots+P_n y=Q \dots\ \dots\ ..\ (1)$$

where, $P_1,\ P_2,\ \dots\ P_n$ are all constants (not functions of x) and Q is function of x or constant but $Q\neq 0$.

The general solution of equation (1) is,

$$y=y_c+y_p$$

where y_c is known as the complementary function (C.F) and y_p is called the particular integral (P.I).

Working Rules for Finding Particular Integral:

1. $\frac{1}{f(D)}x^m=\left[1\pm F(D)\right]^{-1}x^m$.
2. $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$ if $f(a)\neq 0$.
3. $\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax;\ \ or,\ \frac{1}{f(D^2)}\cos ax=\frac{1}{f(-a^2)}\cos ax$ if $f(-a^2)\neq 0$.
4. $\frac{1}{f(D)}e^{ax}V=e^{ax}\frac{1}{f(D+a)}V$ *where, V is a function of x.*

Exceptional case:

1. $\frac{1}{f(D)}e^{ax}=\frac{x}{f'(D)}e^{ax}$ if $f(a)=0$ but $f'(a)\neq 0$.

   Again, if $f'(a)=0$, $f''(a)\neq 0$ then

   $$\frac{1}{f(D)}e^{ax}=\frac{x^2}{f''(D)}e^{ax}.$$

2. $\frac{1}{f(D^2)}\sin ax=\frac{x}{f'(D^2)}\sin ax;\ \ or,\ \frac{1}{f(D^2)}\cos ax=\frac{x}{f'(D^2)}\cos ax$ if $f(-a^2)=0$ but $f'(-a^2)\neq 0$.

   Again if $f'(-a^2)=0$ but $f''(-a^2)\neq 0$ then

$$\frac{1}{f\left(D^2\right)}\sin ax=\frac{x^2}{f''\left(D^2\right)}\sin ax;\quad or,\ \frac{1}{f\left(D^2\right)}\cos ax=\frac{x^2}{f''\left(D^2\right)}\cos ax\ .$$

NOTE:

1. $D=\frac{d}{dx}$ and $\frac{1}{D}=\int dx$.
2. $\left(1+x\right)^{-1}=1-x+x^2-x^3+\cdots\cdots\cdots$
3. $\left(1-x\right)^{-1}=1+x+x^2+x^3+\cdots\cdots\cdots$
4. $\left(1+x\right)^{-2}=1+2x+3x^2+4x^3+\cdots\cdots\cdots$
5. $\left(1-x\right)^{-2}=1-2x+3x^2-4x^3+\cdots\cdots\cdots$
6. $\left(1-x\right)^{-3}=1+3x+6x^2+10x^3+\cdots\cdots\cdots\cdots$

Problem-01: Solve $D^2y+3Dy+2y=4x$.

Solution: Given that,

$$D^2y+3Dy+2y=4x\ \ldots\ \ldots\ \ldots\ (1)$$

Let, $y=e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y+3Dy+2y=0\ \ldots\ \ldots\ \ldots\ (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx}+3me^{mx}+2e^{mx}=0$$

$$\Rightarrow e^{mx}\left(m^2+3m+2\right)=0$$

$$\Rightarrow m^2+3m+2=0\quad ;\quad \text{since } e^{mx}\neq 0$$

$$\Rightarrow m^2+2m+m+2=0$$

$$\Rightarrow m\left(m+2\right)+\left(m+2\right)=0$$

$$\Rightarrow \left(m+2\right)\left(m+1\right)=0$$

$$\therefore\ m+2=0\quad ;\ m+1=0$$

$$\therefore\ m=-2\quad ;\ m=-1$$

The complementary function of (1) is,

$$y_c=c_1e^{-x}+c_2e^{-2x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2+3D+2}(4x)$$

$$= \frac{1}{2\left\{1+\left(\frac{1}{2}D^2+\frac{3}{2}D\right)\right\}}(4x)$$

$$= \frac{1}{2}\left\{1+\left(\frac{1}{2}D^2+\frac{3}{2}D\right)\right\}^{-1}(4x)$$

$$= \frac{1}{2}\left\{1-\left(\frac{1}{2}D^2+\frac{3}{2}D\right)+\left(\frac{1}{2}D^2+\frac{3}{2}D\right)^2-\cdots\cdots\cdots\right\}(4x)$$

$$= \frac{1}{2}\left\{4x-\left(0+\frac{3}{2}.4\right)\right\}$$

$$= 2x-3$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1e^{-x}+c_2e^{-2x}+2x+3$$

where, c_1, c_2 are arbitrary constants.

Problem-02: Solve $D^2y+5Dy+4y=3-2x$.

Solution: Given that,

$$D^2y+5Dy+4y=3-2x \cdots\cdots\cdots (1)$$

Let, $y=e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y+5Dy+4y=0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx}+5me^{mx}+4e^{mx}=0$$

$$\Rightarrow e^{mx}\left(m^2+5m+4\right)=0$$

$$\Rightarrow m^2+5m+4=0 \;\;;\;\; \sin ce\; e^{mx}\neq 0$$

$$\Rightarrow m^2+4m+m+4=0$$

$$\Rightarrow m(m+4)+(m+4)=0$$

$$\Rightarrow (m+4)(m+1)=0$$

$$\therefore\ m+1=0\ ;\ m+4=0$$

$$\therefore\ m=-1\ ;\ m=-4$$

The complementary function of (1) is,

$$y_c = c_1e^{-x} + c_2e^{-4x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2+5D+4}(3-2x)$$

$$= \frac{1}{4\left\{1+\left(\frac{1}{4}D^2+\frac{5}{4}D\right)\right\}}(3-2x)$$

$$= \frac{1}{4}\left\{1+\left(\frac{1}{4}D^2+\frac{5}{4}D\right)\right\}^{-1}(3-2x)$$

$$= \frac{1}{4}\left\{1-\left(\frac{1}{4}D^2+\frac{5}{4}D\right)+\left(\frac{1}{4}D^2+\frac{5}{4}D\right)^2-\cdots\cdots\cdots\right\}(3-2x)$$

$$= \frac{1}{4}\left\{3-2x-\left(0-\frac{5}{2}\right)\right\}$$

$$= \frac{3}{4}-\frac{1}{2}x+\frac{5}{8}$$

$$= \frac{11}{8}-\frac{1}{2}x$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1e^{-x} + c_2e^{-4x} + \frac{11}{8} - \frac{1}{2}x$$

where, c_1, c_2 are arbitrary constants.

Problem-03: Solve $D^2y+2Dy+y=2x+x^2$.

Solution: Given that,

$$D^2y+2Dy+y=2x+x^2 \cdots\cdots\cdots(1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2 y + 2Dy + y = 0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2 e^{mx} + 2me^{mx} + e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 + 2m + 1\right) = 0$$

$$\Rightarrow m^2 + 2m + 1 = 0 \;\; ; \;\; \text{since } e^{mx} \neq 0$$

$$\Rightarrow (m+1)^2 = 0$$

$$\therefore \;\; m = -1\,, -1$$

The complementary function of (1) is,

$$y_c = (c_1 + c_2 x)e^{-x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 + 2D + 1}\left(2x + x^2\right)$$

$$= \frac{1}{1 + \left(D^2 + 2D\right)}\left(2x + x^2\right)$$

$$= \left[1 + \left(D^2 + 2D\right)\right]^{-1}\left(2x + x^2\right)$$

$$= \left[1 - \left(D^2 + 2D\right) + \left(D^2 + 2D\right)^2 - \cdots\cdots\cdots\right]\left(2x + x^2\right)$$

$$= \left[2x + x^2 - \{2 + 2(2 + 2x)\} + 8\right]$$

$$= 2x + x^2 - 2 - 4 - 4x + 8$$

$$= x^2 - 2x + 2$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$$

where, c_1, c_2 are arbitrary constants.

Problem-04: Solve $D^2 y - 6Dy + 9y = 1 + x + x^2$.

Solution: Given that,

$$D^2y - 6Dy + 9y = 1 + x + x^2 \cdots\cdots\cdots \text{ (1)}$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - 6Dy + 9y = 0 \cdots\cdots\cdots \text{ (2)}$$

The auxiliary equation of (2) is,

$$m^2e^{mx} - 6me^{mx} + 9e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 6m + 9\right) = 0$$

$$\Rightarrow m^2 - 6m + 9 = 0 \quad ; \quad \sin ce \ e^{mx} \neq 0$$

$$\Rightarrow (m-3)^2 = 0$$

$$\therefore \quad m = 3, 3$$

The complementary function of (1) is,

$$y_c = (c_1 + c_2x)e^{3x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 6D + 9}\left(1 + x + x^2\right)$$

$$= \frac{1}{9\left[1 + \left(\frac{1}{9}D^2 - \frac{2}{3}D\right)\right]}\left(1 + x + x^2\right)$$

$$= \frac{1}{9}\left[1 + \left(\frac{1}{9}D^2 - \frac{2}{3}D\right)\right]^{-1}\left(1 + x + x^2\right)$$

$$= \frac{1}{9}\left[1 - \left(\frac{1}{9}D^2 - \frac{2}{3}D\right) + \left(\frac{1}{9}D^2 - \frac{2}{3}D\right)^2 - \cdots\cdots\cdots\right]\left(1 + x + x^2\right)$$

$$= \frac{1}{9}\left[1 + x + x^2 - \left\{\frac{2}{9} - \frac{2}{3}(1 + 2x)\right\} + \frac{8}{9}\right]$$

$$= \frac{1}{9}\left[1 + x + x^2 - \frac{2}{9} + \frac{2}{3} + \frac{4}{3}x + \frac{8}{9}\right]$$

$$= \frac{1}{27}\left(3x^2 + 7x + 7\right)$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^{3x} + \frac{1}{27}(3x^2 + 7x + 7)$$

where, c_1, c_2 are arbitrary constants.

Problem-05: Solve $D^4y - 2D^3y + D^2y = x^3$.

Solution: Given that,

$$D^4y - 2D^3y + D^2y = x^3 \cdots\cdots\cdots \ (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^4y - 2D^3y + D^2y = 0 \cdots\cdots\cdots \ (2)$$

The auxiliary equation of (2) is,

$$m^4e^{mx} - 2m^3e^{mx} + m^2e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^4 - 2m^3 + m^2\right) = 0$$

$$\Rightarrow m^4 - 2m^3 + m^2 = 0 \ ; \ \ since \ e^{mx} \neq 0$$

$$\Rightarrow m^2\left(m^2 - 2m + 1\right) = 0$$

$$\Rightarrow m^2(m-1)^2 = 0$$

$$\therefore \ m = 0, \ 0, 1, 1$$

The complementary function of (1) is,

$$y_c = c_1 + c_2x + (c_3 + c_4x)e^x$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^4 - 2D^3 + D^2}\left(x^3\right)$$

$$= \frac{1}{D^2\left[1 + \left(D^2 - 2D\right)\right]}\left(x^3\right)$$

$$= \frac{1}{D^2}\left[1 + \left(D^2 - 2D\right)\right]^{-1}\left(x^3\right)$$

$$= \frac{1}{D^2}\left[x^3 - \left(6x - 6x^2\right) + (-24 + 24x)\right]$$

$$= \frac{1}{D^2}\left(x^3 - 6x + 6x^2 - 24 + 24x\right)$$

$$= \frac{1}{D^2}\left(x^3 + 6x^2 + 18x - 24\right)$$

$$= \frac{1}{20}x^5 + \frac{1}{2}x^4 + 3x^3 - 12x^2$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 + c_2 x + \left(c_3 + c_4 x\right)e^x + \frac{1}{20}x^5 + \frac{1}{2}x^4 + 3x^3 - 12x^2$$

where, $c_1,\ c_2, c_3, c_4$ are arbitrary constants.

Exercise: Try Yourself

1. $D^2y - 2Dy + y = x^2$ **Ans:** $y = \left(c_1 + c_2 x\right)e^x + x^2 + 4x + 6$
2. $D^2y + 4y = x^2 + 3$ **Ans:** $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} + \frac{5}{8}$
3. $D^2y + Dy - 2y = 2\left(1 + x - x^2\right)$ **Ans:** $y = c_1 e^x + c_2 e^{-2x} + x^2$

Problem-06: Solve $D^2y - Dy - 2y = e^x$.

Solution: Given that,

$$D^2y - Dy - 2y = e^x \cdots\cdots\cdots (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - Dy - 2y = 0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2 e^{mx} - me^{mx} - 2e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - m - 2\right) = 0$$

$$\Rightarrow m^2 - m - 2 = 0 \ \ ; \ \ \text{since } e^{mx} \neq 0$$

$$\Rightarrow m^2 - 2m + m - 2 = 0$$

$$\Rightarrow m\left(m-2\right) + \left(m-2\right) = 0$$

$$\Rightarrow \left(m-2\right)\left(m+1\right) = 0$$

$$\therefore \ m = -1, 2$$

The complementary function of (1) is,

$$y_c = c_1 e^{-x} + c_2 e^{2x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - D - 2}\left(e^x\right)$$

$$= \frac{e^x}{1^2 - 1 - 2}$$

$$= -\frac{e^x}{2}$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{2x} - \frac{e^x}{2}$$

where, c_1, c_2 are arbitrary constants.

Problem-07: Solve $D^2 y + 4Dy + 3y = e^{-3x}$.

Solution: Given that,

$$D^2 y + 4Dy + 3y = e^{-3x} \cdots\cdots\cdots (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2 y + 4Dy + 3y = 0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2 e^{mx} + 4me^{mx} + 3e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 + 4m + 3\right) = 0$$

$$\Rightarrow m^2 + 4m + 3 = 0 \;\; ; \;\; \text{since } e^{mx} \neq 0$$

$$\Rightarrow m^2 + 3m + m + 3 = 0$$

$$\Rightarrow m(m+3) + (m+3) = 0$$

$$\Rightarrow (m+3)(m+1) = 0$$

$$\therefore \;\; m = -1, -3$$

The complementary function of (1) is,

$$y_c = c_1 e^{-x} + c_2 e^{-3x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 + 4D + 3}\left(e^{-3x}\right)$$

$$= \frac{x}{2D+4}\left(e^{-3x}\right)$$

$$= \frac{xe^{-3x}}{2(-3)+4}$$

$$= -\frac{1}{2}xe^{-3x}$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2}xe^{-3x}$$

where, c_1, c_2 are arbitrary constants.

Problem-08: Solve $D^3 y + y = 3 + e^{-x} + 5e^{2x}$.

Solution: Given that,

$$D^3 y + y = 3 + e^{-x} + 5e^{2x} \cdots\cdots\cdots \text{ (1)}$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^3 y + y = 0 \cdots\cdots\cdots \text{ (2)}$$

The auxiliary equation of (2) is,

$$m^3 e^{mx} + e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^3 + 1\right) = 0$$

$$\Rightarrow m^3 + 1 = 0 \;\; ; \;\; \sin ce \;\; e^{mx} \neq 0$$

$$\Rightarrow (m+1)\left(m^2 - m + 1\right) = 0$$

$$\therefore \;\; m + 1 = 0 \quad or, \;\; m^2 - m + 1 = 0$$

$$\Rightarrow \;\; m = -1 \;\; or, \;\; m = \frac{1 \pm \sqrt{1-4}}{2}$$

$$=\frac{1\pm\sqrt{-3}}{2}$$

$$=\frac{1\pm\sqrt{3i^2}}{2}$$

$$=\frac{1\pm\sqrt{3}i}{2}$$

$$\therefore\quad m=-1,\frac{1\pm\sqrt{3}i}{2}$$

The complementary function of (1) is,

$$y_c=c_1e^{-x}+e^{x/2}\left\{c_2\cos\left(\frac{\sqrt{3}}{2}x\right)+c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$$

The particular integral of (1) is,

$$y_p=\frac{1}{D^3+1}\left(3+e^{-x}+5e^{2x}\right)$$

$$=\frac{1}{D^3+1}(3)+\frac{1}{D^3+1}\left(e^{-x}\right)+\frac{1}{D^3+1}\left(5e^{2x}\right)$$

$$=\left(1+D^3\right)^{-1}(3)+\frac{x}{3D^2}\left(e^{-x}\right)+\frac{5e^{2x}}{2^3+1}$$

$$=\left(1-D^3+D^6-\cdots\cdots\cdots\right)(3)+\frac{xe^{-x}}{3(-1)^2}+\frac{5e^{2x}}{9}$$

$$=3+\frac{1}{3}xe^{-x}+\frac{5}{9}e^{2x}$$

Therefore, the general solution of equation (1) is,

$$y=y_c+y_p$$

$$=c_1e^{-x}+e^{x/2}\left\{c_2\cos\left(\frac{\sqrt{3}}{2}x\right)+c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}+3+\frac{1}{3}xe^{-x}+\frac{5}{9}e^{2x}$$

where, c_1, c_2, c_3 are arbitrary constants.

Problem-09: Solve $D^3y+3D^2y+3Dy+y=e^{-x}$.

Solution: Given that,

$$D^3y+3D^2y+3Dy+y=e^{-x}\ \cdots\cdots\cdots\ (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^3y + 3D^2y + 3Dy + y = 0 \cdots\cdots\cdots \ (2)$$

The auxiliary equation of (2) is,

$$m^3e^{mx} + 3m^2e^{mx} + 3me^{mx} + e^{mx} = 0$$

$$\Rightarrow e^{mx} \left(m^3 + 3m^2 + 3m + 1\right) = 0$$

$$\Rightarrow m^3 + 3m^2 + 3m + 1 = 0 \ \ ; \ \ \text{since } e^{mx} \neq 0$$

$$\Rightarrow (m+1)^3 = 0$$

$$\therefore \ \ m = -1, -1, -1$$

The complementary function of (1) is,

$$y_c = \left(c_1 + c_2x + c_3x^2\right)e^{-x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^3 + 3D^2 + 3D + 1}\left(e^{-x}\right)$$

$$= \frac{x}{3D^2 + 6D + 3}\left(e^{-x}\right)$$

$$= \frac{x^2}{6D + 6}\left(e^{-x}\right)$$

$$= \frac{x^3}{6}e^{-x}$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= \left(c_1 + c_2x + c_3x^2\right)e^{-x} + \frac{x^3}{6}e^{-x}$$

where, c_1, c_2, c_3 are arbitrary constants.

Problem-10: Solve $2D^3y - 3D^2y + y = e^x + 1$.

Solution: Given that,

$$2D^3y - 3D^2y + y = e^x + 1 \cdots\cdots\cdots \ (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$2D^3y - 3D^2y + y = 0 \cdots\cdots\cdots \; (2)$$

The auxiliary equation of (2) is,

$$2m^3e^{mx} - 3m^2e^{mx} + e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(2m^3 - 3m^2 + 1\right) = 0$$

$$\Rightarrow 2m^3 - 3m^2 + 1 = 0 \;\; ; \;\; \sin ce \; e^{mx} \neq 0$$

$$\Rightarrow 2m^3 - 2m^2 - m^2 + m - m + 1 = 0$$

$$\Rightarrow 2m^2(m-1) - m(m-1) - (m-1) = 0$$

$$\Rightarrow (m-1)\left(2m^2 - m - 1\right) = 0$$

$$\Rightarrow (m-1)\left(2m^2 - 2m + m - 1\right) = 0$$

$$\Rightarrow (m-1)\left\{2m(m-1) + (m-1)\right\} = 0$$

$$\Rightarrow (m-1)(m-1)(2m+1) = 0$$

$$\therefore \;\; m = 1, 1, -\frac{1}{2}$$

The complementary function of (1) is,

$$y_c = \left(c_1 + c_2x\right)e^x + c_3e^{-\frac{x}{2}}$$

The particular integral of (1) is,

$$y_p = \frac{1}{2D^3 - 3D^2 + 1}\left(e^x + 1\right)$$

$$= \frac{x}{6D^2 - 6D}\left(e^x\right) + \frac{1}{2D^3 - 3D^2 + 1}$$

$$= \frac{x^2}{12D - 6}\left(e^x\right) + 1$$

$$= \frac{x^2}{12(1) - 6}\left(e^x\right) + 1$$

$$= \frac{1}{6}x^2e^x + 1$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^x + c_3 e^{-\frac{x}{2}} + \frac{1}{6}x^2 e^x + 1$$

where, c_1, c_2, c_3 are arbitrary constants.

Exercise: Try Yourself

1. $D^2 y - 3Dy + 2y = e^{3x}$ **Ans:** $y = c_1 e^x + c_2 e^{2x} + \frac{1}{2}e^{3x}$
2. $D^3 y - Dy = e^x + e^{-x}$ **Ans:** $y = c_1 + c_2 e^x + c_3 e^{-x} + \frac{x}{2}\left(e^x + e^{-x}\right)$
3. $D^2 y + 2Dy + 2y = 2e^{-x}$ **Ans:** $y = (c_1 \cos x + c_2 \sin x)e^{-x} + 2e^{-x}$
4. $D^2 y + 4Dy + 4y = e^{2x} + e^{-2x}$ **Ans:** $y = (c_1 + c_2 x)e^{-2x} + \frac{e^{2x}}{16} + \frac{1}{2}x^2 e^{-2x}$

Problem-11: Solve $D^2 y + 4y = \sin 3x$.

Solution: Given that,

$$D^2 y + 4y = \sin 3x \cdots\cdots\cdots (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2 y + 4y = 0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2 e^{mx} + 4e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 + 4\right) = 0$$

$$\Rightarrow m^2 + 4 = 0 \quad ; \quad \sin ce\ e^{mx} \neq 0$$

$$\Rightarrow m^2 - (2i)^2 = 0$$

$$\Rightarrow (m + 2i)(m - 2i) = 0$$

$$\therefore \quad m = 2i, -2i$$

The complementary function of (1) is,

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 + 4}(\sin 3x)$$

$$= \frac{1}{-3^2+4}(\sin 3x)$$

$$= \frac{1}{-9+4}(\sin 3x)$$

$$= -\frac{1}{5}\sin 3x$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5}\sin 3x$$

where, c_1, c_2 are arbitrary constants.

Problem-12: Solve $D^2y - 2Dy + 5y = 10\sin x$.

Solution: Given that,

$$D^2y - 2Dy + 5y = 10\sin x \cdots\cdots\cdots \text{ (1)}$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - 2Dy + 5y = 0 \cdots\cdots\cdots \text{ (2)}$$

The auxiliary equation of (2) is,

$$m^2e^{mx} - 2me^{mx} + 5e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 2m + 5\right) = 0$$

$$\Rightarrow m^2 - 2m + 5 = 0 \;\; ; \;\; \sin ce\ e^{mx} \neq 0$$

$$\therefore \quad m = \frac{2 \pm \sqrt{4 - 4\times 5}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{16i^2}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

$$\therefore \quad m = 1 + 2i, 1 - 2i$$

The complementary function of (1) is,

$$y_c = (c_1 \cos 2x + c_2 \sin 2x)e^x$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 2D + 5}(10\sin x)$$

$$= \frac{1}{-1^2 - 2D + 5}(10\sin x)$$

$$= \frac{1}{4 - 2D}(10\sin x)$$

$$= \frac{1}{2(2 - D)}(10\sin x)$$

$$= \frac{(2 + D)}{2\{2^2 - D^2\}}(10\sin x)$$

$$= \frac{(2 + D)}{2\{2^2 - (-1^2)\}}(10\sin x)$$

$$= \frac{1}{10}(20\sin x + 10\cos x)$$

$$= 2\sin x + \cos x$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= (c_1 \cos 2x + c_2 \sin 2x)e^x + 2\sin x + \cos x$$

where, c_1, c_2 are arbitrary constants.

Problem-13: Solve $D^2 y - 8Dy + 16y = 5\cos 3x$.

Solution: Given that,

$$D^2 y - 8Dy + 16y = 5\cos 3x \cdots\cdots\cdots (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2 y - 8Dy + 16y = 0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2 e^{mx} - 8me^{mx} + 16e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2-8m+16\right)=0$$

$$\Rightarrow m^2-8m+16=0 \quad ; \quad \sin ce\ e^{mx}\neq 0$$

$$\Rightarrow (m-4)^2=0$$

$$\therefore \quad m=4,4$$

The complementary function of (1) is,

$$y_c=(c_1+c_2x)e^{4x}$$

The particular integral of (1) is,

$$y_p=\frac{1}{D^2-8D+16}(5\cos 3x)$$

$$=\frac{1}{-3^2-8D+16}(5\cos 3x)$$

$$=\frac{1}{7-8D}(5\cos 3x)$$

$$=\frac{(7+8D)}{7^2-(8D)^2}(5\cos 3x)$$

$$=\frac{(7+8D)}{49-64D^2}(5\cos 3x)$$

$$=\frac{1}{49-64\left(-3^2\right)}(35\cos 3x-120\sin 3x)$$

$$=\frac{1}{625}(35\cos 3x-120\sin 3x)$$

$$=\frac{1}{125}(7\cos 3x-24\sin 3x)$$

Therefore, the general solution of equation (1) is,

$$y=y_c+y_p$$

$$=(c_1+c_2x)e^{4x}+\frac{1}{125}(7\cos 3x-24\sin 3x)$$

where, c_1, c_2 are arbitrary constants.

Problem-14: Solve $D^2y-3Dy+4y=\cos(3x+5)$.

Solution: Given that,

$$D^2y - 3Dy + 4y = \cos(3x+5) \cdots\cdots\cdots \; (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - 3Dy + 4y = 0 \cdots\cdots\cdots \; (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx} - 3me^{mx} + 4e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 3m + 4\right) = 0$$

$$\Rightarrow m^2 - 3m + 4 = 0 \;\; ; \;\; \sin ce \; e^{mx} \neq 0$$

$$\therefore \; m = \frac{3 \pm \sqrt{9-16}}{2}$$

$$= \frac{3 \pm \sqrt{-7}}{2}$$

$$= \frac{3 \pm \sqrt{7}i}{2}$$

$$\therefore \; m = \frac{3+\sqrt{7}i}{2}, \frac{3-\sqrt{7}i}{2}$$

The complementary function of (1) is,

$$y_c = \left[c_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right]e^{\frac{3x}{2}}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 3D + 4}\left[\cos(4x+5)\right]$$

$$= \frac{1}{-4^2 - 3D + 4}\left[\cos(4x+5)\right]$$

$$= \frac{1}{-12 - 3D}\left[\cos(4x+5)\right]$$

$$= -\frac{1}{3(4+D)}\left[\cos(4x+5)\right]$$

$$= -\frac{(4-D)}{3(4^2 - D^2)}\left[\cos(4x+5)\right]$$

$$= -\frac{(4-D)}{3\left\{4^2-\left(-4^2\right)\right\}}\left[\cos(4x+5)\right]$$

$$= -\frac{1}{96}\left[4\cos(4x+5)+4\sin(4x+5)\right]$$

$$= -\frac{1}{24}\left[\cos(4x+5)+\sin(4x+5)\right]$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= \left[c_1\cos\left(\frac{\sqrt{7}}{2}x\right)+c_2\sin\left(\frac{\sqrt{7}}{2}x\right)\right]e^{\frac{3x}{2}} - \frac{1}{24}\left[\cos(4x+5)+\sin(4x+5)\right]$$

where, c_1, c_2 are arbitrary constants.

Problem-15: Solve $D^2y + y = \sin 2x \sin x$.

Solution: Given that,

$$D^2y + y = \sin 2x \sin x \cdots\cdots\cdots \text{ (1)}$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y + y = 0 \cdots\cdots\cdots \text{ (2)}$$

The auxiliary equation of (2) is,

$$m^2e^{mx} + e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2+1\right) = 0$$

$$\Rightarrow m^2 + 1 = 0 \;\; ; \;\; \sin ce \; e^{mx} \neq 0$$

$$\Rightarrow m^2 - i^2 = 0$$

$$\Rightarrow (m+i)(m-i) = 0$$

$$\therefore \;\; m = i, -i$$

The complementary function of (1) is,

$$y_c = c_1\cos x + c_2\sin x$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2+1}\left(\sin 2x \sin x\right)$$

$$= \frac{1}{D^2+1}\left(\frac{1}{2}\times 2\sin 2x \sin x\right)$$

$$= \frac{1}{2}\frac{1}{D^2+1}(\cos x - \cos 3x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}(\cos x) - \frac{1}{D^2+1}(\cos 3x)\right]$$

$$= \frac{1}{2}\left[\frac{x}{2D}(\cos x) - \frac{1}{-3^2+1}(\cos 3x)\right]$$

$$= \frac{1}{2}\left[\frac{xD}{2D^2}(\cos x) + \frac{1}{8}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{xD}{2\left(-1^2\right)}(\cos x) + \frac{1}{8}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{-2}(-\sin x) + \frac{1}{8}\cos 3x\right]$$

$$= \frac{1}{16}(4x\sin x + \cos 3x)$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 \cos x + c_2 \sin x + \frac{1}{16}(4x\sin x + \cos 3x)$$

where, c_1, c_2 are arbitrary constants.

Problem-16: Solve $D^2 y + 4y = \sin^2 x$.

Solution: Given that,

$$D^2 y + 4y = \sin^2 x \cdots\cdots\cdots \;(1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2 y + 4y = 0 \cdots\cdots\cdots \;(2)$$

The auxiliary equation of (2) is,

$$m^2 e^{mx} + 4e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2+4\right) = 0$$

$$\Rightarrow m^2+4=0 \;\; ; \;\; \sin ce\ e^{mx} \neq 0$$

$$\Rightarrow m^2-(2i)^2=0$$

$$\Rightarrow (m+2i)(m-2i)=0$$

$$\therefore \;\; m=2i,-2i$$

The complementary function of (1) is,

$$y_c=c_1\cos 2x+c_2\sin 2x$$

The particular integral of (1) is,

$$y_p=\frac{1}{D^2+4}\left(\sin^2 x\right)$$

$$=\frac{1}{D^2+4}\left(\frac{1}{2}\times 2\sin^2 x\right)$$

$$=\frac{1}{2}\frac{1}{D^2+4}(1-\cos 2x)$$

$$=\frac{1}{2}\left[\frac{1}{4}\frac{1}{\left(1+\frac{D^2}{4}\right)}-\frac{x}{2D}(\cos 2x)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}(1)-\frac{xD}{2D^2}(\cos 2x)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}-\cdots\cdots\right)(1)-\frac{xD}{2\left(-2^2\right)}(\cos 2x)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}+\frac{x}{8}(-2\sin 2x)\right]$$

$$=\frac{1}{2}\left(\frac{1}{4}-\frac{x}{4}\sin 2x\right)$$

$$=\frac{1}{8}(1-x\sin 2x)$$

Therefore, the general solution of equation (1) is,

$$y=y_c+y_p$$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(1 - x \sin 2x)$$

where, c_1, c_2 are arbitrary constants.

Exercise: Try Yourself

1. $D^2y + 3Dy + 2y = \cos 2x$ **Ans:** $y = c_1e^{-x} + c_2e^{-2x} + \frac{1}{20}(3\sin 2x - \cos 2x)$
2. $D^2y - 5Dy + 6y = 100\sin 4x$ **Ans:** $y = c_1e^{3x} + c_2e^{2x} + 4\cos 4x - 2\sin 4x$
3. $D^2y + 4y = \sin 2x$ **Ans:** $y = c_1 \cos 2x + c_2 \sin 2x - \frac{x}{4}\cos 2x$
4. $D^2y + 4y = \cos 2x$ **Ans:** $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4}\sin 2x$
5. $D^2y + y = \sin x$ **Ans:** $y = c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x$
6. $D^2y + y = \cos^2 x$ **Ans:** $y = c_1 \cos x + c_2 \sin x + \frac{1}{2} - \frac{1}{6}\cos 2x$
7. $D^2y - 5Dy + 6y = \sin x + \text{co}\ \ x$ **Ans:** $y = c_1e^{3x} + c_2e^{2x} + \frac{1}{5}\cos x$
8. $D^3y - y = \sin(3x+1)$ **Ans:** $y = c_1e^x + \left\{c_2 \cos\left(\sqrt{3}x/2\right) + c_3 \sin\left(\sqrt{3}x/2\right)\right\}e^{-x/2}$

   $+ 1/730\{27\cos(3x+1) - \sin(3x+1)\}$

Problem-17: Solve $D^2y - 4Dy - 5y = xe^{-x}$.

Solution: Given that,

$$D^2y - 4Dy - 5y = xe^{-x} \cdots\cdots\cdots (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - 4Dy - 5y = 0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx} - 4me^{mx} - 5 = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 4m - 5\right) = 0$$

$$\Rightarrow m^2 - 4m - 5 = 0 \;\; ; \;\; \text{since } e^{mx} \neq 0$$

$$\Rightarrow m^2 - 5m + m - 5 = 0$$

$$\Rightarrow m(m-5) + (m-5) = 0$$

$$\Rightarrow (m+1)(m-5) = 0$$

$\therefore \quad m=-1,5$

The complementary function of (1) is,

$$y_c = c_1e^{-x} + c_2e^{5x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2-4D-5}\left(xe^{-x}\right)$$

$$= e^{-x}\frac{1}{(D-1)^2-4(D-1)-5}(x)$$

$$= e^{-x}\frac{1}{D^2-2D+1-4D+4-5}(x)$$

$$= e^{-x}\frac{1}{D^2-6D}(x)$$

$$= -\frac{1}{6}e^{-x}\frac{1}{D\left(1-D/6\right)}(x)$$

$$= -\frac{1}{6}e^{-x}\frac{1}{D}\left(1-D/6\right)^{-1}(x)$$

$$= -\frac{1}{6}e^{-x}\frac{1}{D}\left(1+D/6+D^2/36^2+\cdots\cdots\cdots\right)(x)$$

$$= -\frac{1}{6}e^{-x}\frac{1}{D}\left(x+1/6\right)$$

$$= -\frac{1}{6}e^{-x}\left(\frac{x^2}{2}+x/6\right)$$

$$= -\frac{1}{36}e^{-x}\left(x+3x^2\right)$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1e^{-x} + c_2e^{5x} - \frac{1}{36}e^{-x}\left(x+3x^2\right)$$

where, c_1, c_2 are arbitrary constants.

Problem-18: Solve $D^2y - y = (x+3)e^{2x}$.

Solution: Given that,

$$D^2y - y = (x+3)e^{2x} \cdots\cdots\cdots \ (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - y = 0 \cdots\cdots\cdots \ (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx} - e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2-1\right) = 0$$

$$\Rightarrow m^2 - 1 = 0 \quad ; \quad \sin ce \ e^{mx} \neq 0$$

$$\Rightarrow (m+1)(m-1) = 0$$

$$\therefore \ m = -1, 1$$

The complementary function of (1) is,

$$y_c = c_1e^{-x} + c_2e^{x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2-1}(x+3)e^{2x}$$

$$= e^{2x}\frac{1}{(D+2)^2-1}(x+3)$$

$$= e^{2x}\frac{1}{D^2+4D+4-1}(x+3)$$

$$= e^{2x}\frac{1}{D^2+4D+3}(x+3)$$

$$= \frac{1}{3}e^{2x}\frac{1}{\left[1+\left(\frac{4}{3}D+\frac{1}{3}D^2\right)\right]}(x+3)$$

$$= \frac{1}{3}e^{2x}\left[1+\left(\frac{4}{3}D+\frac{1}{3}D^2\right)\right]^{-1}(x+3)$$

$$= \frac{1}{3}e^{2x}\left[1-\left(\frac{4}{3}D+\frac{1}{3}D^2\right)+\left(\frac{4}{3}D+\frac{1}{3}D^2\right)^2-\cdots\cdots\cdots\right](x+3)$$

$$= \frac{1}{3}e^{2x}\left[x+3-\left(\frac{4}{3}+0\right)+0\right]$$

$$=\frac{1}{9}e^{2x}\left(3x+5\right)$$

Therefore, the general solution of equation (1) is,

$$y=y_c+y_p$$

$$=c_1e^{-x}+c_2e^{x}+\frac{1}{9}e^{2x}\left(3x+5\right)$$

where, $c_1,\ c_2$ are arbitrary constants.

Problem-19: Solve $D^3y-2Dy+4y=e^x\cos x$.

Solution: Given that,

$$D^3y-2Dy+4y=e^x\cos x \cdots\cdots\cdots \text{ (1)}$$

Let, $y=e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^3y-2Dy+4y=0 \cdots\cdots\cdots \text{ (2)}$$

The auxiliary equation of (2) is,

$$m^3e^{mx}-2me^{mx}+4e^{mx}=0$$

$$\Rightarrow e^{mx}\left(m^3-2m+4\right)=0$$

$$\Rightarrow m^3-2m+4=0\ \ ;\ \ \sin ce\ e^{mx}\neq 0$$

$$\Rightarrow m^3+2m^2-2m^2-4m+2m+4=0$$

$$\Rightarrow m^2\left(m+2\right)-2m\left(m+2\right)+2\left(m+2\right)=0$$

$$\Rightarrow\left(m+2\right)\left(m^2-2m+2\right)=0$$

$$\therefore\ m+2=0\ \ or,\ m^2-2m+2=0$$

$$\Rightarrow\ m=-2\ \ or,\ m=\frac{2\pm\sqrt{4-4\times 2}}{2}$$

$$=\frac{2\pm\sqrt{-4}}{2}$$

$$=\frac{2\pm\sqrt{4i^2}}{2}$$

$$=\frac{2\pm 2i}{2}$$

$$=1\pm i$$

$$\therefore \quad m=-2,1+i,\,1-i$$

The complementary function of (1) is,

$$y_c = c_1 e^{-2x} + \left(c_2 \cos x + c_3 \sin x\right)e^x$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^3 - 2D + 4}\left(e^x \cos x\right)$$

$$= e^x \frac{1}{\left(D+1\right)^3 - 2\left(D+1\right)+4}\left(\cos x\right)$$

$$= e^x \frac{1}{D^3 + 3D^2 + 3D + 1 - 2D - 2 + 4}\left(\cos x\right)$$

$$= e^x \frac{1}{D^3 + 3D^2 + D + 3}\left(\cos x\right)$$

$$= e^x \frac{x}{3D^2 + 6D + 1}\left(\cos x\right)$$

$$= e^x \frac{x}{3\left(-1^2\right) + 6D + 1}\left(\cos x\right)$$

$$= e^x \frac{x}{-2 + 6D}\left(\cos x\right)$$

$$= -\frac{1}{2} e^x \frac{x}{\left(1-3D\right)}\left(\cos x\right)$$

$$= -\frac{1}{2} e^x \frac{x\left(1+3D\right)}{\left\{1-\left(3D\right)^2\right\}}\left(\cos x\right)$$

$$= -\frac{1}{2} e^x \frac{x\left(1+3D\right)}{\left\{1-9\left(-1^2\right)\right\}}\left(\cos x\right)$$

$$= -\frac{1}{20} x e^x \left(\cos x - 3\sin x\right)$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 e^{-2x} + \left(c_2 \cos x + c_3 \sin x\right)e^x - \frac{1}{20} x e^x \left(\cos x - 3\sin x\right)$$

where, c_1, c_2, c_3 are arbitrary constants.

Problem-20: Solve $D^2y - 2Dy + 2y = e^x \sin x$.

Solution: Given that,

$$D^2y - 2Dy + 2y = e^x \sin x \cdots\cdots\cdots \quad (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - 2Dy + 2y = e^x \sin x \cdots\cdots\cdots \quad (2)$$

The auxiliary equation of (2) is,

$$m^2 e^{mx} - 2me^{mx} + 2e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 2m + 2\right) = 0$$

$$\Rightarrow m^2 - 2m + 2 = 0 \;\; ; \;\; \sin ce \; e^{mx} \neq 0$$

$$\therefore \;\; m = \frac{2 \pm \sqrt{4 - 4 \times 2}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm \sqrt{4i^2}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

$$\therefore \;\; m = 1 + i, 1 - i$$

The complementary function of (1) is,

$$y_c = \left(c_1 \cos x + c_2 \sin x\right) e^x$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 2D + 2}\left(e^x \sin x\right)$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}(\sin x)$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 2}(\sin x)$$

$$= e^{x}\frac{1}{D^{2}+1}(\sin x)$$

$$= e^{x}\frac{x}{2D}(\sin x)$$

$$= e^{x}\frac{xD}{2D^{2}}(\sin x)$$

$$= e^{x}\frac{xD}{2(-1^{2})}(\sin x)$$

$$= -\frac{1}{2}xe^{x}(\cos x)$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= (c_1\cos x + c_2\sin x)e^{x} - \frac{1}{2}xe^{x}(\cos x)$$

where, c_1, c_2 are arbitrary constants.

Problem-21: Solve $D^2y + 3Dy + 2y = e^{2x}\sin x$.

Solution: Given that,

$$D^2y + 3Dy + 2y = e^{2x}\sin x \cdots\cdots\cdots \text{(1)}$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y + 3Dy + 2y = 0 \cdots\cdots\cdots \text{(2)}$$

The auxiliary equation of (2) is,

$$m^2e^{mx} + 3me^{mx} + 2e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 + 3m + 2\right) = 0$$

$$\Rightarrow m^2 + 3m + 2 = 0 \;\; ; \;\; \sin ce\ e^{mx} \neq 0$$

$$\Rightarrow m^2 + 2m + m + 2 = 0$$

$$\Rightarrow m(m+2) + (m+2) = 0$$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\therefore \;\; m = -1, -2$$

The complementary function of (1) is,

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2+3D+2}\left(e^{2x}\sin x\right)$$

$$= e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}(\sin x)$$

$$= e^{2x}\frac{1}{D^2+4D+4+3D+6+2}(\sin x)$$

$$= e^{2x}\frac{1}{D^2+7D+12}(\sin x)$$

$$= e^{2x}\frac{1}{-1^2+7D+12}(\sin x)$$

$$= e^{2x}\frac{1}{11+7D}(\sin x)$$

$$= e^{2x}\frac{(11-7D)}{(11)^2-(7D)^2}(\sin x)$$

$$= e^{2x}\frac{(11-7D)}{121-49D^2}(\sin x)$$

$$= e^{2x}\frac{(11-7D)}{121-49\left(-1^2\right)}(\sin x)$$

$$= \frac{1}{170}(11\sin x - 7\cos x)e^{2x}$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{170}(11\sin x - 7\cos x)e^{2x}$$

where, c_1, c_2 are arbitrary constants.

Problem-22: Solve $D^2y - 2Dy + y = x\sin x$.

Solution: Given that,

$$D^2y - 2Dy + y = x\sin x \cdots\cdots\cdots \ (1)$$

Let, $y=e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y-2Dy+y=0\cdots\cdots\cdots\ (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx}-2me^{mx}+e^{mx}=0$$

$$\Rightarrow e^{mx}\left(m^2-2m+1\right)=0$$

$$\Rightarrow m^2-2m+1=0\ \ ;\ \ \sin ce\ e^{mx}\neq 0$$

$$\Rightarrow (m-1)^2=0$$

$$\therefore\ \ m=1,1$$

The complementary function of (1) is,

$$y_c=(c_1+c_2x)e^x$$

The particular integral of (1) is,

$$y_p=\frac{1}{D^2-2D+1}(x\sin x)$$

$$=Imaginary\ \ Part\ \ of\left[\frac{1}{D^2-2D+1}\left(xe^{ix}\right)\right]$$

$$=I.\ P.\ of\ \left[e^{ix}\frac{1}{(D+i)^2-2(D+i)+1}(x)\right]$$

$$=I.\ P.\ of\ \left[e^{ix}\frac{1}{D^2+2Di-1-2D-2i+1}(x)\right]$$

$$=I.\ P.\ of\ \left[e^{ix}\frac{1}{D^2+2Di-2D-2i}(x)\right]$$

$$=I.\ P.\ of\ \left[-\frac{e^{ix}}{2i}\frac{1}{\left\{1-\frac{1}{2i}\left(D^2+2Di-2D\right)\right\}}(x)\right]$$

$$=I.P.\ of\ \left[\frac{ie^{ix}}{2}\left\{1-\frac{1}{2i}\left(D^2+2Di-2D\right)\right\}^{-1}(x)\right]$$

$$=I.P.\ of\ \left[\frac{ie^{ix}}{2}\left\{1+\frac{1}{2i}\left(D^2+2Di-2D\right)+\frac{1}{4i^2}\left(D^2+2Di-2D\right)^2+\cdots\cdots\right\}(x)\right]$$

$$= I.P.\ of\ \left[\frac{ie^{ix}}{2}\left\{x+\frac{1}{2i}(2i-2)\right\}\right]$$

$$= I.P.\ of\ \left[\frac{ie^{ix}}{2}\{x+(1+i)\}\right]$$

$$= I.P.\ of\ \left[\frac{i}{2}(\cos x + i\sin x)\{(x+1)+i\}\right]$$

$$= I.P.\ o\ \left[f\frac{i}{2}\{(x+1)\cos x + i\cos x + i(x+1)\sin x - \sin x\}\right]$$

$$= I.P.\ of\ \left[\frac{1}{2}\{i(x+1)\cos x - \cos x - (x+1)\sin x - i\sin x\}\right]$$

$$= \frac{1}{2}\{(x+1)\cos x - \sin x\}$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= (c_1 + c_2 x)e^x + \frac{1}{2}\{(x+1)\cos x - \sin x\}$$

where, c_1, c_2 are arbitrary constants.

Problem-23: Solve $D^2y + Dy = x\cos x$.

Solution: Given that,

$$D^2y + Dy = x\cos x \cdots\cdots\cdots (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y + Dy = 0 \cdots\cdots\cdots (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx} + me^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 + m\right) = 0$$

$$\Rightarrow m^2 + m = 0\ \ ;\ \ \sin ce\ e^{mx} \neq 0$$

$$\Rightarrow m(m+1) = 0$$

$$\therefore\ \ m = 0, -1$$

The complementary function of (1) is,

$$y_c = c_1 + c_2 e^{-x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 + D}\left(x \cos x\right)$$

$$= \text{Re}\,al\ \ Part\ \ of \left[\frac{1}{D^2 + D}\left(xe^{ix}\right)\right]$$

$$= \text{R}.\ P.\ of \left[e^{ix} \frac{1}{\left(D+i\right)^2 + \left(D+i\right)}\left(x\right)\right]$$

$$= \text{R}.\ P.\ of \left[e^{ix} \frac{1}{D^2 + 2iD + i^2 + \left(D+i\right)}\left(x\right)\right]$$

$$= \text{R}.\ P.\ of \left[e^{ix} \frac{1}{D^2 + 2iD - 1 + D + i}\left(x\right)\right]$$

$$= \text{R}.\ P.\ of \left[e^{ix} \frac{1}{D^2 + \left(2i+1\right)D + \left(i-1\right)}\left(x\right)\right]$$

$$= \text{R}.\ P.\ of \left[\frac{1}{\left(i-1\right)} e^{ix} \frac{1}{1 + \left\{\frac{1}{\left(i-1\right)} D^2 + \frac{\left(2i+1\right)}{\left(i-1\right)} D\right\}}\left(x\right)\right]$$

$$= \text{R}.\ P.\ of \left[\frac{1}{\left(i-1\right)} e^{ix} \left\{1 + \left(\frac{1}{\left(i-1\right)} D^2 + \frac{\left(2i+1\right)}{\left(i-1\right)} D\right)\right\}^{-1} \left(x\right)\right]$$

$$= \text{R}.\ P.\ o \left[\frac{1}{\left(i-1\right)} e^{ix} f \left\{1 - \left(\frac{1}{\left(i-1\right)} D^2 + \frac{\left(2i+1\right)}{\left(i-1\right)} D\right) + \left(\frac{1}{\left(i-1\right)} D^2 + \frac{\left(2i+1\right)}{\left(i-1\right)} D\right)^2 - \cdots\cdots\right\}\left(x\right)\right]$$

$$= \text{R}.\ P.\ of \left[\frac{1}{\left(i-1\right)} e^{ix} \left\{x - \frac{\left(2i+1\right)}{\left(i-1\right)}\right\}\right]$$

$$= \text{R}.\ P.\ of \left[\frac{1}{\left(i-1\right)^2} e^{ix} \left\{\left(i-1\right)x - \left(2i+1\right)\right\}\right]$$

$$= \text{R}.\,P.\,of\left[\frac{1}{-1+1-2i}(\cos x + i\sin x)\{ix - x - 2i - 1\}\right]$$

$$= \text{R}.\,P.\,of\left[-\frac{1}{2i}(\cos x + i\sin x)\{i(x-2)-(x+1)\}\right]$$

$$= \text{R}.\,P.\,o\,f\left[\frac{i}{2}\{i(x-2)\cos x - (x+1)\cos x - (x-2)\sin x - i(x+1)\sin x\}\right]$$

$$= \frac{1}{2}\{-(x-2)\cos x + (x+1)\sin x\}$$

$$= \frac{1}{2}\{(2-x)\cos x + (x+1)\sin x\}$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 + c_2 e^{-x} + \frac{1}{2}\{(2-x)\cos x + (x+1)\sin x\}$$

where, c_1, c_2 are arbitrary constants.

Problem-24: Solve $D^4 y - y = x\sin x$.

Solution: Given that,

$$D^4 y - y = x\sin x \cdots\cdots\cdots \text{ (1)}$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^4 y - y = 0 \cdots\cdots\cdots \text{ (2)}$$

The auxiliary equation of (2) is,

$$m^4 e^{mx} - e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^4 - 1\right) = 0$$

$$\Rightarrow m^4 - 1 = 0 \;\; ; \;\; \sin ce\ e^{mx} \neq 0$$

$$\Rightarrow \left(m^2 - 1\right)\left(m^2 + 1\right) = 0$$

$$\Rightarrow (m+1)(m-1)\left(m^2 - i^2\right) = 0$$

$$\Rightarrow (m+1)(m-1)(m+i)(m-i) = 0$$

$$\therefore \quad m = -1, 1, i, -i$$

The complementary function of (1) is,

$$y_c = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^4 - 1}\left(x \sin x\right)$$

$$= Imaginary\ \ Part\ \ of \left[\frac{1}{D^4 - 1}\left(x e^{ix}\right)\right]$$

$$= I.\ P.\ of\ \left[e^{ix} \frac{1}{\left(D + i\right)^4 - 1}\left(x\right)\right]$$

$$= I.\ P.\ of\ \left[e^{ix} \frac{1}{D^4 + 4D^3 i + 6D^2 i^2 + 4Di^3 + i^4 - 1}\left(x\right)\right]$$

$$\left[\sin ce, \left(a + b\right)^n = a^n + {}^n c_1 a^{n-1} b + {}^n c_2 a^{n-2} b^2 + \cdots\cdots + b^n\right]$$

$$= I.\ P.\ of\ \left[e^{ix} \frac{1}{D^4 + 4D^3 i - 6D^2 - 4Di + 1 - 1}\left(x\right)\right]$$

$$= I.\ P.\ of\ \left[e^{ix} \frac{1}{D^4 + 4D^3 i - 6D^2 - 4Di}\left(x\right)\right]$$

$$= I.\ P.\ of\ \left[-\frac{e^{ix}}{4Di} \frac{1}{\left\{1 - \left(\frac{1}{4i} D^3 + D^2 + \frac{3i}{2} D\right)\right\}}\left(x\right)\right]$$

$$= I.\ P.\ of\ \left[\frac{ie^{ix}}{4D}\left\{1 - \left(\frac{1}{4i} D^3 + D^2 + \frac{3i}{2} D\right)\right\}^{-1}\left(x\right)\right]$$

$$= I.\ P.\ of\ \left[\frac{ie^{ix}}{4D}\left\{1 + \left(\frac{1}{4i} D^3 + D^2 + \frac{3i}{2} D\right) + \left(\frac{1}{4i} D^3 + D^2 + \frac{3i}{2} D\right)^2 + \cdots\cdots\right\}\left(x\right)\right]$$

$$= I.\ P.\ of\ \left[\frac{ie^{ix}}{4D}\left(x + \frac{3i}{2}\right)\right]$$

$$= I.\ P.\ of\ \left[\frac{ie^{ix}}{4}\left(\frac{x^2}{2} + \frac{3xi}{2}\right)\right]$$

$$= I.\ P.\ of\ \left[\frac{i}{4}(\cos x + i\sin x)\left(\frac{x^2}{2}+\frac{3xi}{2}\right)\right]$$

$$= I.\ P.\ of\ \left[\frac{1}{4}(i\cos x - \sin x)\left(\frac{x^2}{2}+\frac{3xi}{2}\right)\right]$$

$$= I.\ P.\ of\ \left[\frac{1}{4}\left(i\frac{x^2}{2}\cos x - \frac{3x}{2}\cos x - \frac{x^2}{2}\sin x - \frac{3xi}{2}\sin x\right)\right]$$

$$= \frac{1}{4}\left(\frac{x^2}{2}\cos x - \frac{3x}{2}\sin x\right)$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1e^{-x} + c_2e^{x} + c_3\cos x + c_4\sin x + \frac{1}{4}\left(\frac{x^2}{2}\cos x - \frac{3x}{2}\sin x\right)$$

where, c_1, c_2 are arbitrary constants.

Problem-25: Solve $D^2y - y = x^2\cos x$.

Solution: Given that,

$$D^2y - y = x^2\cos x \cdots\cdots\cdots \ (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - y = 0 \cdots\cdots\cdots \ (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx} - e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 1\right) = 0$$

$$\Rightarrow (m+1)(m-1) = 0 \ \ ; \ \ \sin ce\ e^{mx} \neq 0$$

$$\therefore\ m = 1, -1$$

The complementary function of (1) is,

$$y_c = c_1e^{x} + c_2e^{-x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 1}\left(x^2 \cos x\right)$$

$$= \text{Re}\,al\;\;Part\;\;of\left[\frac{1}{D^2-1}\left(x^2 e^{ix}\right)\right]$$

$$= \text{R}.\;P.\;of\left[e^{ix}\frac{1}{\left(D+i\right)^2-1}\left(x^2\right)\right]$$

$$= \text{R}.\;P.\;of\left[e^{ix}\frac{1}{D^2+2iD+i^2-1}\left(x^2\right)\right]$$

$$= \text{R}.\;P.\;of\left[e^{ix}\frac{1}{D^2+2iD-1-1}\left(x^2\right)\right]$$

$$= \text{R}.\;P.\;of\left[e^{ix}\frac{1}{D^2+2iD-2}\left(x^2\right)\right]$$

$$= \text{R}.\;P.\;of\left[-\frac{1}{2}e^{ix}\frac{1}{1-\left(\frac{1}{2}D^2+iD\right)}\left(x^2\right)\right]$$

$$= \text{R}.\;P.\;of\left[-\frac{1}{2}e^{ix}\left\{1-\left(\frac{1}{2}D^2+iD\right)\right\}^{-1}\left(x^2\right)\right]$$

$$= \text{R}.\;P.\;of\left[-\frac{1}{2}e^{ix}\left\{1+\left(\frac{1}{2}D^2+iD\right)+\left(\frac{1}{2}D^2+iD\right)^2+\cdots\cdots\right\}\left(x^2\right)\right]$$

$$= \text{R}.\;P.\;of\left[-\frac{1}{2}e^{ix}\left\{x^2+\left(1+2ix\right)-2\right\}\right]$$

$$= \text{R}.\;P.\;of\left[-\frac{1}{2}\left(\cos x+i\sin x\right)\left\{x^2+1+2ix-2\right\}\right]$$

$$= \text{R}.\;P.\;of\left[-\frac{1}{2}\left(\cos x+i\sin x\right)\left\{\left(x^2-1\right)+2ix\right\}\right]$$

$$= \text{R}.\;P.\;of\left[-\frac{1}{2}\left\{\left(x^2-1\right)\cos x+2ix\cos x+i\left(x^2-1\right)\sin x-2x\sin x\right\}\right]$$

$$= \frac{1}{2}\left\{2x\sin x-\left(x^2-1\right)\cos x\right\}$$

$$= x\sin x-\frac{1}{2}\left(x^2-1\right)\cos x$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1e^x + c_2e^{-x} + x\sin x - \frac{1}{2}\left(x^2 - 1\right)\cos x$$

where, c_1, c_2 are arbitrary constants.

Problem-26: Solve $D^2y - y = xe^x \sin x$.

Solution: Given that,

$$D^2y - y = xe^x \sin x \cdots\cdots\cdots \ (1)$$

Let, $y = e^{mx}$ be the trial solution of the corresponding homogeneous equation,

$$D^2y - y = 0 \cdots\cdots\cdots \ (2)$$

The auxiliary equation of (2) is,

$$m^2e^{mx} - e^{mx} = 0$$

$$\Rightarrow e^{mx}\left(m^2 - 1\right) = 0$$

$$\Rightarrow (m+1)(m-1) = 0 \ \ ; \ \ \sin ce \ e^{mx} \neq 0$$

$$\therefore \ \ m = 1, -1$$

The complementary function of (1) is,

$$y_c = c_1e^x + c_2e^{-x}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 1}\left(x^2e^x \sin x\right)$$

$$y_p = e^x \frac{1}{(D+1)^2 - 1}\left(x^2 \sin x\right)$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 1}\left(x^2 \sin x\right)$$

$$= e^x \frac{1}{D^2 + 2D}\left(x^2 \sin x\right)$$

$$= Imaginary \ \ Part \ \ of\left[e^x \frac{1}{D^2 + 2D}\left(x^2e^{ix}\right)\right]$$

$$= I.\ P.\ of \left[e^x e^{ix} \frac{1}{(D+i)^2 + 2(D+i)} \left(x^2\right) \right]$$

$$= I.\ P.\ of \left[e^{(1+i)x} \frac{1}{D^2 + 2Di + i^2 + 2D + 2i} \left(x^2\right) \right]$$

$$= I.\ P.\ of \left[e^x e^{ix} \frac{1}{D^2 + 2(1+i)D - 1 + 2i} \left(x^2\right) \right]$$

$$= I.\ P.\ of \left[\frac{1}{(2i-1)} e^x e^{ix} \frac{1}{1 + \left\{D^2 + 2(1+i)D\right\}} \left(x^2\right) \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^x e^{ix} \left[1 + \left\{D^2 + 2(1+i)D\right\}\right]^{-1} \left(x^2\right) \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^{(1+i)x} \left[1 - \left\{D^2 + 2(1+i)D\right\} + \left\{D^2 + 2(1+i)D\right\}^2 - \cdots\cdots \right] \left(x^2\right) \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^x (\cos x + i \sin x) \left[x^2 - \left\{2 + 4(1+i)x\right\} + 8(1+i)^2 \right] \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^x (\cos x + i \sin x) \left\{ x^2 - 2 - 4x - 4ix + 16i \right\} \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^x (\cos x + i \sin x) \left\{ \left(x^2 - 4x - 2\right) - 4i(x-4) \right\} \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^x (\cos x + i \sin x) \left\{ \left(x^2 - 4x - 2\right) - 4i(x-4) \right\} \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^x \left\{ \left(x^2 - 4x - 2\right)\cos x - 4i(x-4)\cos x + i\left(x^2 - 4x - 2\right)\sin x + 4(x-4)\sin x \right\} \right]$$

$$= I.\ P.\ of \left[-\frac{(2i+1)}{5} e^x \left\{ \left(x^2 - 4x - 2\right)\cos x - 4i(x-4)\cos x + i\left(x^2 - 4x - 2\right)\sin x + 4(x-4)\sin x \right\} \right]$$

$$-\frac{(2i+1)}{5} e^x \left\{ 2\left(x^2 - 4x - 2\right)\cos x - 4i(x-4)\cos x + i\left(x^2 - 4x - 2\right)\sin x + 4(x-4)\sin x \right\}$$

Therefore, the general solution of equation (1) is,

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{-x} + x\sin x - \frac{1}{2}\left(x^2 - 1\right)\cos x$$

where, c_1, c_2 are arbitrary constants.

Exercise: Try Yourself

1. $D^2 y + 4y = x\sin x$ **Ans:** $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{9}\left(3x\sin x - 2\cos x\right)$

Linear Differential Equations with variables Coefficients

An equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots\cdots + P_n y = Q \cdots\cdots\cdots \; (1)$$

where, P_1, P_2, $\cdots\cdots P_n$ are constants and Q is function of x or constant, is called the linear differential equation with variables coefficients.

NOTE: If we put $x = e^t \;\; or,\; t = \ln x$, then the equation (1) is transformed into an equation with constant coefficients changing the independent variable from x to t as,

$$\frac{dt}{dx} = \frac{1}{x}$$

Now $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\Rightarrow x\frac{dy}{dx} = Dy \quad ; taking \;\; D = \frac{d}{dt}$$

Again, $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}.\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}.\frac{1}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = D(D-1)y \quad ;taking\ D = \frac{d}{dt}$$

Similarly, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

... ...

$$x^n\frac{d^ny}{dx^n} = D(D-1)(D-2)\cdots\cdots\cdots(D-n+1)\ y$$

From (1) we get,

$$\left[\{D(D-1)(D-2)\cdots\cdots\cdots(D-n+1)\} + P_1\{D(D-1)(D-2)\cdots\cdots\cdots(D-n+2)\}\right.$$
$$+\cdots\cdots + P_n y = Q\cdots\cdots(2)$$

The equation (2) is a linear differential equation with constant coefficients.

Problem-01: Solve $x^2\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} + 25y = 0$

Solution: Given that, $x^2\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} + 25y = 0 \cdots\cdots\cdots (1)$

Putting $x = e^t\ and\ D = \frac{d}{dt}$ in equation (1) we get,

$$D(D-1)y + 9Dy + 25y = 0$$

$$\Rightarrow D^2y - Dy + 9Dy + 25y = 0$$

$$\Rightarrow D^2y + 8Dy + 25y = 0\cdots\cdots\cdots(2)$$

Let, $y = e^{mt}$ be the trial solution of the equation (2)

Then the auxiliary equation of (2) is,

$$m^2 e^{mt} + 8me^{mt} + 25e^{mt} = 0$$

$$\Rightarrow e^{mt}\left(m^2 + 8m + 25\right) = 0$$

$$\Rightarrow m^2 + 8m + 25 = 0 \ \ ; \ \ \sin ce\ e^{mt} \neq 0$$

$$\therefore\ m = \frac{-8 \pm \sqrt{64-100}}{2}$$

$$= \frac{-8 \pm \sqrt{-36}}{2}$$

$$= \frac{-8 \pm 6i}{2}$$

$$= -4 \pm 3i$$

$$\therefore\ m = -4 \pm 3i$$

The general solution of (1) is,

$$y = \left(c_1 \cos 3t + c_2 \sin 3t\right) e^{-4t}$$
$$= \left[c_1 \cos\left(3\ln x\right) + c_2 \sin\left(3\ln x\right)\right] x^{-4}$$
$$= \frac{1}{x^4}\left[c_1 \cos\left(3\ln x\right) + c_2 \sin\left(3\ln x\right)\right]$$

where, c_1, c_2 are arbitrary constants.

Problem-02: Solve $x^2 \dfrac{d^2 y}{dx^2} + y = 3x^2$

Solution: Given that, $x^2 \dfrac{d^2 y}{dx^2} + y = 3x^2 \ \cdots\cdots\cdots (1)$

Putting $x = e^t \ and\ D = \dfrac{d}{dt}$ in equation (1) we get,

$$D\left(D-1\right) y + y = 3e^{2t}$$

$$\Rightarrow D^2 y - Dy + y = 3e^{2t} \cdots\cdots\cdots\left(2\right)$$

Let, $y = e^{mt}$ be the trial solution of the corresponding homogeneous equation

$D^2 y - Dy + y = 0 \cdots\cdots\cdots\left(3\right)$

Then the auxiliary equation of (3) is,

$$m^2e^{mt} - me^{mt} + e^{mt} = 0$$

$$\Rightarrow e^{mt}\left(m^2 - m + 1\right) = 0$$

$$\Rightarrow m^2 - m + 1 = 0 \quad ; \quad \sin ce\ e^{mt} \neq 0$$

$$\therefore\ m = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore\ m = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

The complementary function of (1) is,

$$y_c = \left[c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] e^{t/2}$$

$$= \left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] e^{\ln x^{\frac{1}{2}}}$$

$$= \sqrt{x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - D + 1}\left(3e^{2t}\right)$$

$$= \frac{1}{2^2 - 2 + 1}\left(3e^{2t}\right)$$

$$= e^{2t}$$

$$= x^2$$

Therefore the general solution is,

$$y = y_c + y_p$$

$$= \sqrt{x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] + x^2$$

where, c_1, c_2 are arbitrary constants.

Problem-03: Solve $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$

Solution: Given that, $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4 \cdots\cdots\cdots (1)$

Putting $x=e^t$ *and* $D=\frac{d}{dt}$ in equation (1) we get,

$$D(D-1)y-2Dy-4y=e^{4t}$$

$$\Rightarrow D^2y-3Dy-4y=e^{4t}\cdots\cdots\cdots(2)$$

Let, $y=e^{mt}$ be the trial solution of the corresponding homogeneous equation

$$D^2y-3Dy-4y=0\cdots\cdots\cdots(3)$$

Then the auxiliary equation of (3) is,

$$m^2e^{mt}-3me^{mt}-4e^{mt}=0$$

$$\Rightarrow e^{mt}\left(m^2-3m-4\right)=0$$

$$\Rightarrow m^2-3m-4=0 \;\; ; \;\; \sin ce\ e^{mt}\neq 0$$

$$\Rightarrow m^2-4m+m-4=0$$

$$\Rightarrow (m-4)+(m-4)=0$$

$$\Rightarrow (m+1)(m-4)=0$$

$$\therefore \;\; m=-1,\ 4$$

The complementary function of (1) is,

$$y_c=c_1e^{-t}+c_2e^{4t}$$

$$=c_1x^{-1}+c_2x^4$$

$$=\frac{c_1}{x}+c_2x^4$$

The particular integral of (1) is,

$$y_p=\frac{1}{D^2-3D-4}\left(e^{4t}\right)$$

$$=\frac{t}{2D-3}\left(e^{4t}\right)$$

$$= \frac{t}{2.4-3}\left(e^{4t}\right)$$

$$= \frac{t}{5}\left(e^{4t}\right)$$

$$= \frac{1}{5}x^4 \ln x$$

Therefore the general solution is,

$$y = y_c + y_p$$

$$= \frac{c_1}{x} + c_2 x^4 + \frac{1}{5}x^4 \ln x$$

where, c_1, c_2 are arbitrary constants.

Problem-04: Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$

Solution: Given that, $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2 \cdots\cdots\cdots (1)$

Putting $x = e^t$ and $D = \frac{d}{dt}$ in equation (1) we get,

$$D(D-1)y - 3Dy + 4y = 2e^{2t}$$

$$\Rightarrow D^2 y - 4Dy + 4y = 2e^{2t} \cdots\cdots\cdots(2)$$

Let, $y = e^{mt}$ be the trial solution of the corresponding homogeneous equation

$$D^2 y - 4Dy + 4y = 0 \cdots\cdots\cdots(3)$$

Then the auxiliary equation of (3) is,

$$m^2 e^{mt} - 4me^{mt} + 4e^{mt} = 0$$

$$\Rightarrow e^{mt}\left(m^2 - 4m + 4\right) = 0$$

$$\Rightarrow m^2 - 4m + 4 = 0 \ \ ; \ \ \text{since } e^{mt} \neq 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\therefore \ m = 2,\ 2$$

The complementary function of (1) is,

$$y_c = (c_1 + c_2 t)e^{2t}$$

$$= x^2\left(c_1 + c_2 \ln x\right)$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 4D + 4}\left(2e^{2t}\right)$$

$$= \frac{t}{2D-4}\left(2e^{2t}\right)$$

$$= \frac{t^2}{2}\left(2e^{2t}\right)$$

$$= t^2 e^{2t}$$

$$= \left(\ln x\right)^2 x^2$$

Therefore the general solution is,

$$y = y_c + y_p$$

$$= x^2\left(c_1 + c_2 \ln x\right) + \left(\ln x\right)^2 x^2$$

where, c_1, c_2 are arbitrary constants.

Problem-05: Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \ln x$

Solution: Given that, $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \ln x \cdots\cdots\cdots (1)$

Putting $x = e^t$ *and* $D = \dfrac{d}{dt}$ in equation (1) we get,

$$D\left(D-1\right)y - Dy - 3y = te^{2t}$$

$$\Rightarrow D^2 y - 2Dy - 3y = te^{2t} \cdots\cdots\cdots\left(2\right)$$

Let, $y = e^{mt}$ be the trial solution of the corresponding homogeneous equation

$$D^2 y - 2Dy - 3y = 0 \cdots\cdots\cdots\left(3\right)$$

Then the auxiliary equation of (3) is,

$$m^2 e^{mt} - 2me^{mt} - 3e^{mt} = 0$$

$$\Rightarrow e^{mt}\left(m^2 - 2m - 3\right) = 0$$

$$\Rightarrow m^2 - 2m - 3 = 0 \;\; ; \;\; \text{since } e^{mt} \neq 0$$

$$\Rightarrow m^2 - 3m + m - 3 = 0$$

$$\Rightarrow m(m-3) + (m-3) = 0$$

$$\Rightarrow (m+1)(m-3) = 0$$

$$\therefore \quad m = -1,\ 3$$

The complementary function of (1) is,

$$y_c = c_1 e^{-t} + c_2 e^{3t}$$
$$= c_1 x^{-1} + c_2 x^3$$
$$= \frac{c_1}{x} + c_2 x^3$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 - 2D - 3}\left(te^{2t}\right)$$

$$= e^{2t}\frac{1}{(D+2)^2 - 2(D+2) - 3}(t)$$

$$= e^{2t}\frac{1}{D^2 + 4D + 4 - 2D - 4 - 3}(t)$$

$$= e^{2t}\frac{1}{D^2 + 2D - 3}(t)$$

$$= -\frac{e^{2t}}{3}\frac{1}{\left[1 - \left(\frac{D^2}{3} + \frac{2}{3}D\right)\right]}(t)$$

$$= -\frac{e^{2t}}{3}\left[1 - \left(\frac{D^2}{3} + \frac{2}{3}D\right)\right]^{-1}(t)$$

$$= -\frac{e^{2t}}{3}\left[1 + \left(\frac{D^2}{3} + \frac{2}{3}D\right) + \left(\frac{D^2}{3} + \frac{2}{3}D\right)^2 + \cdots\cdots\cdots\right](t)$$

$$= -\frac{e^{2t}}{3}\left[t + \left(\frac{D^2}{3} + \frac{2}{3}D\right)t + \left(\frac{D^2}{3} + \frac{2}{3}D\right)^2 t + \cdots\cdots\cdots\right]$$

$$= -\frac{e^{2t}}{3}\left[t + \left(0 + \frac{2}{3}\right) + 0\right]$$

$$= -\frac{e^{2t}}{3}\left(t + \frac{2}{3}\right)$$

$$= -\frac{x^2}{3}\left(\ln x + \frac{2}{3}\right)$$

Therefore the general solution is,

$$y = y_c + y_p$$

$$= \frac{c_1}{x} + c_2 x^3 - \frac{x^2}{3}\left(\ln x + \frac{2}{3}\right)$$

where, c_1, c_2 are arbitrary constants.

Problem-06: Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \sin x$

Solution: Given that, $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \sin x \cdots\cdots\cdots (1)$

Putting $x = e^t$ *and* $D = \frac{d}{dt}$ in equation (1) we get,

$$D(D-1)y + 4Dy + 2y = e^t + \sin e^t$$

$$\Rightarrow D^2y + 3Dy + 2y = e^t + \sin e^t \cdots\cdots\cdots (2)$$

Let, $y = e^{mt}$ be the trial solution of the corresponding homogeneous equation

$D^2y + 3Dy + 2y = 0 \cdots\cdots\cdots (3)$

Then the auxiliary equation of (3) is,

$$m^2e^{mt} + 3me^{mt} + 2e^{mt} = 0$$

$$\Rightarrow e^{mt}\left(m^2 + 3m + 2\right) = 0$$

$$\Rightarrow m^2 + 3m + 2 = 0 \;\; ; \;\; \sin ce\ e^{mt} \neq 0$$

$$\Rightarrow m^2 + 2m + m + 2 = 0$$

$$\Rightarrow m(m+2) + (m+2) = 0$$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\therefore \;\; m = -1, -2$$

The complementary function of (1) is,

$$y_c = c_1 e^{-t} + c_2 e^{-2t}$$
$$= c_1 x^{-1} + c_2 x^{-2}$$
$$= \frac{c_1}{x} + \frac{c_2}{x^2}$$

The particular integral of (1) is,

$$y_p = \frac{1}{D^2 + 3D + 2}\left(e^t + \sin e^t\right)$$

$$= \frac{1}{D^2 + 3D + 2}\left(e^t\right) + \frac{1}{D^2 + 3D + 2}\left(\sin e^t\right)$$

$$= \frac{1}{1^2 + 3.1 + 2}\left(e^t\right) + \frac{1}{(D+2)(D+1)}\left(\sin e^t\right)$$

$$= \frac{e^t}{6} + \frac{1}{(D+2)(D+1)}\left(\sin e^t\right)$$

Now let, $\frac{1}{(D+1)}\left(\sin e^t\right) = u$

$$\Rightarrow (D+1)u = \sin e^t$$

$$\Rightarrow \frac{du}{dt} + u = \sin e^t$$

which is linear equation

Therefore, $I.F = e^{\int dt} = e^t$

$$\therefore \quad e^t \frac{du}{dt} + e^t u = e^t \sin e^t$$

$$or, \ \frac{d}{dt}\left(e^t u\right) = e^t \sin e^t$$

Integrating,

$$e^t u = \int e^t \sin e^t dt$$

$$= -\cos e^t$$

$$\therefore \ u = -e^{-t} \cos e^t$$

Again, $\frac{1}{(D+2)(D+1)}\left(\sin e^t\right) = \frac{1}{(D+2)} u = \frac{1}{(D+2)}\left(-e^{-t} \cos e^t\right) = v \quad (say)$

$$\therefore \ \frac{1}{(D+2)}\left(-e^{-t}\cos e^{t}\right)=v$$

$$or,\ (D+2)v=-e^{-t}\cos e^{t}$$

$$or,\ \frac{dv}{dt}+2v=-e^{-t}\cos e^{t}$$

which is also a linear equation

Therefore, $I.F=e^{\int 2dt}=e^{2t}$

$$\therefore e^{2t}\frac{dv}{dt}+2ve^{2t}=-e^{t}\cos e^{t}$$

$$or,\ \frac{d}{dt}\left(ve^{2t}\right)=-e^{t}\cos e^{t}$$

Integrating,

$$ve^{2t}=-\int e^{t}\cos e^{t}dt$$

$$=-\sin e^{t}$$

$$\therefore v=-\frac{1}{e^{2t}}\sin e^{t}$$

$$=-\frac{1}{x^{2}}\sin x$$

Therefore the general solution is,

$$y=y_c+y_p$$

$$=\frac{c_1}{x}+\frac{c_2}{x^2}-\frac{1}{x^2}\sin x$$

where, $c_1,\ c_2$ are arbitrary constants.

Problem-07: Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^{x}$

Solution: Given that, $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^{x}\cdots\cdots\cdots(1)$

Putting $x=e^{t}\ and\ D=\frac{d}{dt}$ in equation (1) we get,

$$D(D-1)y+4Dy+2y=e^{e^t}$$

$$\Rightarrow D^2y+3Dy+2y=e^{e^t}\cdots\cdots\cdots(2)$$

Let, $y=e^{mt}$ be the trial solution of the corresponding homogeneous equation

$$D^2y+3Dy+2y=0\cdots\cdots\cdots(3)$$

Then the auxiliary equation of (3) is,

$$m^2e^{mt}+3me^{mt}+2e^{mt}=0$$

$$\Rightarrow e^{mt}\left(m^2+3m+2\right)=0$$

$$\Rightarrow m^2+3m+2=0\;\;;\;\; \sin ce\; e^{mt}\neq 0$$

$$\Rightarrow m^2+2m+m+2=0$$

$$\Rightarrow m(m+2)+(m+2)=0$$

$$\Rightarrow (m+1)(m+2)=0$$

$$\therefore\;\; m=-1,\,-2$$

The complementary function of (1) is,

$$y_c=c_1e^{-t}+c_2e^{-2t}$$
$$=c_1x^{-1}+c_2x^{-2}$$
$$=\frac{c_1}{x}+\frac{c_2}{x^2}$$

The particular integral of (1) is,

$$y_p=\frac{1}{D^2+3D+2}\left(e^{e^t}\right)$$

$$=\frac{1}{(D+1)(D+2)}\left(e^{e^t}\right)$$

$$=\left[\frac{1}{D+1}-\frac{1}{D+2}\right]\left(e^{e^t}\right)$$

$$=\frac{1}{D+1}\left(e^{e^t}\right)-\frac{1}{D+2}\left(e^{e^t}\right)$$

Let , $\frac{1}{D+1}\left(e^{e^t}\right)=u$

$$or,\ (D+1)u = \left(e^{e^t}\right)$$

$$or,\ \frac{du}{dt} + u = e^{e^t}$$

which is linear equation

Therefore, $I.F = e^{\int dt} = e^t$

$$\therefore\quad e^t\frac{du}{dt} + e^t u = e^t.e^{e^t}$$

$$or,\ \frac{d}{dt}\left(e^t u\right) = e^t.e^{e^t}$$

Integrating,

$$e^t u = \int e^t.e^{e^t}dt \qquad ;\ as\ e^t = z$$

$$= e^z$$

$$= e^{e^t}$$

$$= e^x$$

$$\therefore\ \ u = e^{-t}.e^x$$

$$= x^{-1}.e^x$$

$$= \frac{e^x}{x}$$

Again, $\frac{1}{(D+2)}\left(e^{e^t}\right) = v \quad (say)$

$$or,\ (D+2)v = e^{e^t}$$

$$or,\ \frac{dv}{dt} + 2v = e^{e^t}$$

which is also a linear equation

Therefore, $I.F = e^{\int 2dt} = e^{2t}$

$$\therefore\ e^{2t}\frac{dv}{dt} + 2ve^{2t} = e^{2t}.e^{e^t}$$

$$or,\ \frac{d}{dt}\left(ve^{2t}\right) = e^{2t}.e^{e^t}$$

Integrating,

$$ve^{2t} = \int e^{2t}.e^{e^t}\,dt$$

$$= \int e^t.e^t.e^{e^t}\,d$$

$$= \int ze^z dz \qquad ;as\ e^t = z$$

$$= ze^z - e^z$$

$$= xe^x - e^x$$

$$\therefore\ v = e^{-2t}\left(xe^x - e^x\right)$$

$$= x^{-2}\left(xe^x - e^x\right)$$

$$= \frac{e^x}{x} - \frac{e^x}{x^2}$$

$$\therefore\quad P.I = \frac{e^x}{x} - \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)$$

$$= \frac{e^x}{x^2}$$

Therefore the general solution is,

$$y = y_c + y_p$$

$$= \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}$$

where, c_1, c_2 are arbitrary constants.

Exercise: Try Yourself:

01: Solve $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$ $\qquad Ans: y = c_1x^{-1} + c_2x^2$

02: Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$ $\qquad Ans: y = c_1x^3 + c_2x^{-3}$

03: Solve $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x^4$ $\qquad Ans: y = \left(c_1 + c_2\ln x\right)x^{-2} + \frac{1}{36}x^4$

04: Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\ln x$ $\qquad Ans: y = \left(c_1 + c_2\ln x\right)x + 2\ln x$