Formation of partial differential equation:

Problems

(1)Form the partial differential equation by eliminating the arbitrary constants from $z = ax + by + a^2 + b^2$.

Solution:

Given $z = ax + by + a^2 + b^2$ .... (1)

Here we have two arbitrary constants a & b.

Differentiating equation (1) partially with respect to x and y respectively we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \qquad \text{.................. (2)}$$

$$\frac{\partial z}{\partial y} = b \Rightarrow q = a \qquad \text{.................. (3)}$$

Substitute (2) and (3) in (1) we get

$z = px + qy + p^2 + q^2$, which is the required partial differential equation.

Problem-02: Eliminate arbitrary constants from the equation, $z = (x-a)^2 + (y-b)^2$.

Solution: Given that, $z = (x-a)^2 + (y-b)^2$ $\cdots(1)$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = 2(x-a)$$

$$or, \left(\frac{\partial z}{\partial x}\right)^2 = 4(x-a)^2 \qquad [Squaring] \qquad \cdots(2)$$

And differentiating (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y} = 2(y-b)$$

$$or, \left(\frac{\partial z}{\partial y}\right)^2 = 4(y-b)^2 \qquad [Squaring] \qquad \cdots(3)$$

Adding equations (1) and (2), we get

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4(x-a)^2+4(y-b)^2$$

$$=4\left\{(x-a)^2+(y-b)^2\right\}$$

$$=4z \qquad \text{[using eq.(1)]}$$

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4z$$

which is the required partial differential equation.

Problem-03: Find the differential equation of all spheres of radius λ, having centre in the *xy* - plane.

Solution: From the coordinate geometry of three dimensions, the equation of any sphere of radius λ, having centre (*h, k, 0*) in the *xy* – plane is,

$$(x-h)^2+(y-k)^2+z^2=\lambda^2 \qquad \cdots(1)$$

Differentiating (1) partially with respect to *x*, we get

$$2(x-h)+2z\frac{\partial z}{\partial x}=0$$

$$or,(x-h)=-z\frac{\partial z}{\partial x}$$

$$or,(x-h)^2=z^2\left(\frac{\partial z}{\partial x}\right)^2 \qquad [Squaring] \qquad \cdots(2)$$

And differentiating (1) partially with respect to *y*, we get

$$2(y-k)+2z\frac{\partial z}{\partial y}=0$$

$$or,(y-k)=-z\frac{\partial z}{\partial y}$$

$$or,(y-k)^2=z^2\left(\frac{\partial z}{\partial y}\right)^2 \qquad [Squaring] \qquad \cdots(3)$$

From equations (1), (2) and (3), we get

$$z^2\left(\frac{\partial z}{\partial x}\right)^2+z^2\left(\frac{\partial z}{\partial y}\right)^2+z^2=\lambda^2$$

which is the required partial differential equation.

Problem-04: Eliminate the arbitrary constants a, b and c from the relation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: Given that, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $\cdots(1)$

Differentiating (1) partially with respect to x, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$or,\ c^2x+a^2z\frac{\partial z}{\partial x}=0 \qquad \cdots(2)$$

Again, differentiating (2) partially with respect to x, we get

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \qquad \cdots(3)$$

From (2), we get

$$c^2=-\frac{a^2z}{x}\frac{\partial z}{\partial x}$$

Putting this value of c^2 in (4), we obtain

$$-\frac{a^2z}{x}\frac{\partial z}{\partial x}+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0$$

$$or,\ -\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0$$

$$or,\ -z\frac{\partial z}{\partial x}+x\left(\frac{\partial z}{\partial x}\right)^2+zx\frac{\partial^2 z}{\partial x^2}=0$$

$$or,\ zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0$$

which is the required partial differential equation.

Problem-05: Form partial differential equation by eliminating constant A and p from $z=Ae^{pt}\sin px$.

Solution: Given that, $z=Ae^{pt}\sin px$ $\cdots(1)$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x} = Ape^{pt} \cos px \qquad \cdots(2)$$

Again, differentiating (2) partially with respect to x, we get

$$\frac{\partial^2 z}{\partial x^2} = -Ap^2 e^{pt} \sin px \qquad \cdots(3)$$

Similarly, differentiating (1) partially with respect to t, we get

$$\frac{\partial z}{\partial t} = Ape^{pt} \sin px \qquad \cdots(4)$$

Again, differentiating (4) partially with respect to t, we get

$$\frac{\partial^2 z}{\partial t^2} = Ap^2 e^{pt} \sin px \qquad \cdots(5)$$

Adding (3) and (5), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = -Ap^2 e^{pt} \sin px + Ap^2 e^{pt} \sin px$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is the required partial differential equation.

6. Form a partial differential equation by eliminating arbitrary constants a and b from

$z = (x+a)^2 + (y+b)^2$

Ans:

Given $z = (x+a)^2 + (y+b)^2 \qquad \ldots\ldots\ (1)$

$$p = \frac{\partial z}{\partial x} = 2(x+a) \qquad \cdots\cdots(2)$$

$$q = \frac{\partial z}{\partial y} = 2(y+b) \qquad \cdots\cdots(3)$$

Substituting (2) & (3) in (1), we get $z = \frac{p^2}{4} + \frac{q^2}{4}$

(7)Find the singular integral of the partial differential equation $z = px + qy + p^2 - q^2$.

Ans:

The complete integral is

$$z = ax + by + a^2 - b^2.$$

$$\frac{\partial z}{\partial a} = x + 2a = 0 \Rightarrow a = -\frac{x}{2}$$

$$\frac{\partial z}{\partial b} = y - 2b = 0 \Rightarrow b = \frac{y}{2}$$

Therefore

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4} = -\frac{x^2}{4} + \frac{y^2}{4}$$

$$\Rightarrow y^2 - x^2 = 4z$$

8. Find the PDE of all planes having equal intercepts on the x and y axis.

Solution:

The equation of such plane is

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1 \qquad \ldots\ldots\ldots (1)$$

Partially differentiating (1) with respect to 'x' and 'y' we get

$$\frac{1}{a} + \frac{p}{b} = 0 \Rightarrow p = -\frac{b}{a}$$

$$\frac{1}{a} + \frac{q}{b} = 0 \Rightarrow q = -\frac{b}{a}$$

From (2) and (3), we get

$$p = q$$

(9)Form a partial differential equation by eliminating the arbitrary constants a and b from $z = ax^n + by^n$.

Ans:

$$Given \quad z = ax^n + by^n. \qquad \ldots\ldots\ldots (1)$$

Partially differentiating with respect to 'x' and 'y' we get

$$p = \frac{\partial z}{\partial x} = a.nx^{n-1} \quad \Rightarrow a = \frac{p}{nx^{n-1}}$$
$$q = \frac{\partial z}{\partial y} = b.ny^{n-1} \quad \Rightarrow b = \frac{q}{ny^{n-1}} \qquad \ldots\ldots\ldots (2)$$

Substituting (2) in (1) we get

$$z = \frac{p}{nx^{n-1}} x^n + \frac{q}{ny^{n-1}} y^n$$
$$z = \frac{1}{n}(px + qy)$$

This is the required PDE.

10. Form a partial differential equation by eliminate the arbitrary function f from $z = f\left(\frac{xy}{z}\right)$.

Ans:

Given: $z = f\left(\frac{xy}{z}\right)$.

$$p = f'\left(\frac{xy}{z}\right) .. \frac{zy - xy.p}{z^2} \qquad \ldots\ldots\ldots\ldots(1)$$
$$q = f'\left(\frac{xy}{z}\right) .. \frac{zx - xy.q}{z^2} \qquad \ldots\ldots\ldots\ldots(2)$$

From (1), we get

$$f'\left(\frac{xy}{z}\right) = \frac{pz^2}{zy - xyp} \qquad \ldots\ldots\ldots\ldots(3)$$

Substituting (3) in(2), we get

$$p = \frac{pz^2}{zy - xyp} \cdot \frac{zy - xy.p}{z^2}$$

Exercise: Try yourself: Find the partial differential equation by eliminating constants from the following relations:

1. $z = Ae^{-p^2 t} \cos px \quad$ (*constants A , p*)
2. $ax^2 + by^2 + cz^2 = 1 \quad$ (*constants a , b*)
3. $z = ax + by + cxy \quad$ (*constants a , b, c*)
4. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad$ (*constants a , b*)