Partial Differential Equations With Constant Coefficients:

We know that the general form of a linear partial differential equation

$$A_n \frac{\partial^n z}{\partial x^n} + A_{n-1} \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_{n-2} \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + A_1 \frac{\partial^n z}{\partial y^n} = f(x, y) \qquad \ldots (1)$$

Where the coefficients A_n, A_{n-1}, A_{n-2}, . . . , A_1 are constants or functions of x and y. If A_n, A_{n-1}, A_{n-2}, . . . , A_1 are all constants, then (1) is called a linear partial differential equation with constant coefficients.

We denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by $D\ (or\ D_x)$ and $D'\left(or\ D_y\right)$ respectively.

Therefore (1) can be written as

$$[A_n D^n + A_{n-1} D^{n-1} D' + A_{n-2} D^{n-2} D'^2 + \ \ldots \ + A_1 D'^n] z = f(x, y) \qquad \ldots (2)$$

or $$\varphi(D, D') z = f(x, y)$$

The complementary function of (2) is given by

$$[A_n D^n + A_{n-1} D^{n-1} D' + A_{n-2} D^{n-2} D'^2 + \cdots + A_1 D'^n] z = 0 \qquad \ldots (3)$$

or $\varphi(D, D') z = 0$

Let $z = F(y + mx)$ be the part of the solution

$$Dz = \frac{\partial z}{\partial x} = mF'(y + mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2 F''(y + mx)$$

.. ..

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n F^n (y + mx)$$

And

$$D' z = \frac{\partial z}{\partial y} = F'(y + mx)$$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = F''(y + mx)$$

... ...

$$D'^n z = \frac{\partial^n z}{\partial y^n} = F^n(y + mx)$$

Substitute these values in (3), we get

$$[A_n m^n + A_{n-1} m^{n-1} + A_{n-2} m^{n-2} + \ . \ . \ . \ + A_1] F^{(n)}(y + mx) = 0$$

which is true if $'m'$ is a root of the equation

If $m_1, m_2,$ $m_n,$ are distinct roots, then complementary functions is

$$z = \varphi_1(y + m_1 x) + \varphi_2(y + m_2 x) + \ . \ . \ . \ + \varphi_n(y + m_n x)$$

where $\varphi_1, \varphi_2, \ . \ . \ .,$ φ_n are arbitrary functions.

$$\therefore \varphi(D, D')z = 0$$

we replace D by m and D' by 1 to get the auxiliary equation from which we get roots.

Linear partial differential equations with constant coefficients

Homogenous and Non homogenous linear equations with constant coefficients: A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a linear partial differential equation. The general form of such an equation is

$$\left[A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial y \partial x^{n-1}} + A_1 \frac{\partial^n z}{\partial y^2 \partial x^{n-2}} + \ . \ . \ . \ + A_n \frac{\partial^n z}{\partial y^n}\right] + \left[B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial y \partial x^{n-2}} + B_1 \frac{\partial^{n-1} z}{\partial y^2 \partial x^{n-3}} + \ . \ . \ . \ + B_n \frac{\partial^{n-1} z}{\partial y^{n-1}}\right] + \left[M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}\right] + N_0 z = f(x, y) \qquad ...(1)$$

where the coefficients $A_0, A_1, \ . \ . \ . \ A_n, B_0, B_1, \ . \ . \ . \ , B_{n-1}, M_0, M_1$ and N_0 are all constants, then (1) is called a linear partial differential equation with constant coefficients.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by D and D' respectively.

2.1. Short method for finding the P.I. in certain cases of F(D,D') z = f(x, y)

2.1.a. Short method I. when f(x, y) is of the form Ø (ax + by)

Ex.1. Solve $(D^2 + 3DD' + 2D'^2)z = x + y$

Sol. The Auxiliary equation of the given equation is

$m^2 + 3m + 2 = 0$ giving $m = -1, -2$

therefore C.F. = $\emptyset_1(y-x) + \emptyset_2(y-2x)$, $\emptyset_1$, $\emptyset_2$ being arbitrary functions

Now P.I. $= \frac{1}{D2 + 3DD' + 2D'2}(x + y)$

$= \frac{1}{1^2+3.1.1+2.1^2} \iint v dv dv$, where v= x+y

$= \int \left(\frac{v^2}{2}\right) dv = \frac{1}{6}\frac{v^3}{6} = \frac{1}{36}(x + y)^3$

Hence the required general solution is z = C.F. + P.I.

or $z = \emptyset_1(y-x) + \emptyset_2(y-2x) + \frac{1}{36}(x + y)^3$

Ex. 2. Solve $(2D^2 - 5DD' + 2D'^2)\, z = 24\,(y - x)$

Sol: Try yourself

Ex.3. Solve $(D^2 + 3DD' + 2D'^2)z = 2x + 3\,y$

Sol: Try yourself

2.1.b.Short method II.

When f(x,y) is of the form $x^m y^n$ or a rational integral

Ex.1. Solve $(D^2 - a^2D'^2)z = x$

Sol. Here auxiliary equation is $m^2 - a^2 = 0$ so that m = a, -a

Therefore C.F. $= \emptyset_1(y + ax) + \emptyset_2(y - ax)$, ...(1)

$\emptyset_1$, $\emptyset_2$ being arbitrary functions.

Now P.I. $= \frac{1}{D^2 - a^2 D'^2}(x) = \frac{1}{D^2\left[1 - a^2\left(\frac{D'^2}{D^2}\right)\right]}(x)$

$= \frac{1}{D2}\left[1 - a^2\left(\frac{D'^2}{D^2}\right)\right]^{-1}(x)$

$= \frac{1}{D2}[1 + a^2(D'^2/D^2) + \ldots]x$

$=\frac{1}{D2}$x

$=\frac{x^3}{6}$

Hence the required solution is z = C.F. + P.I.

$$Z = \emptyset_1(y + a\,x) + \emptyset_2(y - a\,x) + \frac{x^3}{6}$$

Exercise: 1.$2r + 5s + 2t = 0$

Sol: It is a second order pole with constant coefficients, we have

$$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$$

or $\quad (2D^2 + 5DD' + 2D'^2)z = 0$

Now the auxiliary equations is given by

$$(2m^2 + 5m + 2) = 0$$

$\Rightarrow m = -\frac{1}{2}$, -2

Therefore the complementary function is $z = \varphi_1\left(y - \frac{1}{2}x\right) + \varphi_2(y - 2x)$

which is required solution.

Exercise: 2.$r = a^2 t$

Sol: Try Yourself (Ans: $z = \varphi_1(y + ax) + \varphi_2(y - ax)$)

Exercise: 3.$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial^2 y} = 0$

Sol: It is a third order pole with constant coefficients, we have

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial^2 y} = 0$$

or $\qquad (D^3 - 3D^2 D' + 2DD'^2)z = 0$

Now the auxiliary equations is given by

$$m^3 - 3m^2 + 2m = 0$$

$$\Rightarrow m(m^2 - 3m + 2) = 0$$

$$\Rightarrow m(m-1)(m-2) = 0$$

$\Rightarrow m = 0,\ 1\ ,\ 2$

Therefore the complementary function is $z = \varphi_1(y) + \varphi_2(y + x) + \varphi_3(y + 2x)$

Which is required solution.

Exercise: 4. $\frac{\partial^3 z}{\partial x^3} - 6\frac{\partial^3 z}{\partial x^2 \partial y} + 11\frac{\partial^3 z}{\partial x\, \partial^2 y} - 6\frac{\partial^3 z}{\partial y^3} = 0$

Sol: Try yourself

Exercise: 5. $25r - 40s + 16t = 0$

Sol: Try yourself

Exercise: 6. $(D^4 - D'^4)z = 0$

Sol: The auxiliary equation is given by

$$m^4 - 1 = 0$$

$$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow (m-1)(m+1)(m-i)(m+i) = 0$$

$$\Rightarrow m = 1, \quad -1,\ i, \quad -i$$

Therefore the complementary function is

$$z = \varphi_1(y + x) + \varphi_2(y - x) + \varphi_3(y + ix) + \varphi_4(y - ix)$$

which is required solution.

Exercise: 7.$(D^3 - 4D^2D' + 4DD'^2)z = 0$

Sol: Try yourself.

Exercise: 8.$(D^2 - 2DD' + D'^2)z = 12xy$

Sol: The auxiliary equations corresponding to these linear system of equations is given by

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)(m-1) = 0$$

$$\Rightarrow m = 1, \qquad 1$$

Therefore the complementary function is

$$z = f_1(y+x) + xf_2(y+x)$$

Also, P. I. $= \frac{1}{(D-D')^2} 12xy$

$$= \frac{12}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} xy$$

$$= \frac{12}{D^2}\left[1 - 2\frac{D}{D'} + \left(\frac{D}{D'}\right)^2\right]^{-1} xy$$

$$= \frac{12}{D^2}\left[1 - \left\{-2\frac{D}{D'} + \left(\frac{D}{D'}\right)^2\right\} + \quad \right] xy$$

$$= \frac{12}{D^2}\left(xy + \frac{2}{D}x\right)$$

$$= \frac{12}{D}\left(\frac{x^2y}{2} + \frac{x^3}{3}\right)$$

$$= 12\left(\frac{x^3y}{6} + \frac{x^4}{12}\right)$$

$$= 2x^3y + x^4$$

Therefore the complete solution is z=C. F. + P. I.

i.e., $z = f_1(y+x) + xf_2(y+x) + 2x^3y + x^4$

Exercise: 9. $(2D^2 - 5DD' + 2D'^2)z = 24(x - y)$

Sol: Try yourself.

Exercise: 10. $(D^3 - D'^3)z = x^3y^3$

Sol: The auxiliary equations is

$$m^3 - 1 = 0$$

$$\Rightarrow (m - 1)(m^2 + m + 1) = 0$$

$$\Rightarrow m = 1, \ \frac{-1 + i\sqrt{3}}{2}, \ \frac{-1 - i\sqrt{3}}{2}$$

Complementary function is $z = \varphi_1(y + x) + \varphi_2\left(y + \frac{-1+i\sqrt{3}}{2}x\right) + \varphi_3\left(y + \frac{-1-i\sqrt{3}}{2}x\right)$

P. I. $= \frac{1}{D^3 - D'^3} x^3y^3$

$= \frac{1}{D^3\left[1-\left(\frac{D'}{D}\right)^3\right]} x^3y^3$

$= \frac{1}{D^3}\left[1 - \left(\frac{D'}{D}\right)^3\right]^{-1} x^3y^3$

$= \frac{1}{D^3}\left[1 + \left(\frac{D'}{D}\right)^3 + \left(\frac{D'}{D}\right)^6 + \ . \ . \ . \ \right]x^3y^3$

$= \frac{1}{D^3}\left[x^3y^3 + \left(\frac{D'}{D}\right)^3 x^3y^3 + \left(\frac{D'}{D}\right)^6 x^3y^3 + \ . \ . \ . \ \right]$

$= \frac{1}{D^3}\left[x^3y^3 + \frac{D'^2}{D^3} 3x^3y^2 + 0 + \ . \ . \ . \ \right]$

$= \frac{1}{D^3}\left[x^3y^3 + \frac{D'}{D^3} 6x^3y\right]$

$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3} 6x^3\right]$

$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^2} 6\frac{x^4}{4}\right]$

$$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D}6\frac{x^5}{20}\right]$$

$$= \frac{1}{D^3}\left[x^3y^3 + \frac{x^6}{20}\right]$$

$$= \frac{1}{D^2}\left[\frac{x^4y^3}{4} + \frac{x^7}{140}\right]$$

$$= \frac{1}{D}\left[\frac{x^5y^3}{20} + \frac{x^8}{1120}\right]$$

$$= \left[\frac{x^6y^3}{120} + \frac{x^9}{10080}\right]$$

The complete solution is C.F. + P.I.

$$z = \varphi_1(y + x) + \varphi_2\left(y + \frac{-1 + i\sqrt{3}}{2}x\right) + \varphi_3\left(y + \frac{-1 - i\sqrt{3}}{2}x\right) + \frac{x^6y^3}{120} + \frac{x^9}{10080}$$

Exercise: Find the real function $'v'$ of x and y reducing to zero when $y = 0$ and satisfying $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -4\pi(x^2 + y^2)$

Sol: We have to find the P. I. only

$$\text{P. I} = \frac{1}{D^2+D'^2}\left[-4\pi(x^2 + y^2)\right]$$

$$= \frac{1}{D^2\left(1+\frac{D'^2}{D^2}\right)}\left[-4\pi(x^2 + y^2)\right]$$

$$= \frac{-4\pi}{D^2}\left[1 + \frac{D'^2}{D^2}\right]^{-1}(x^2 + y^2)$$

$$= \frac{-4\pi}{D^2}\left[1 - \frac{D'^2}{D^2} + \left(\frac{D'^2}{D^2}\right)^2 + \ . \ . \ . \ \right](x^2 + y^2)$$

$$= \frac{-4\pi}{D^2}\left[(x^2 + y^2) - \frac{D'^2}{D^2}(x^2 + y^2) + 0\ \right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2 + y^2) - \frac{D'}{D^2}(2y)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2 + y^2) - \frac{1}{D^2}(2)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2)-\frac{1}{D}(2x)\right]$$

$$= \frac{-4\pi}{D^2}[(x^2+y^2)-(x^2)]$$

$$= \frac{-4\pi}{D}[xy^2]$$

$$= -4\pi\left[\frac{x^2y^2}{2}\right]$$

$$= -2\pi x^2y^2$$

Problem:

1.

Solve: $(D^2+3DD'-4D'^2)z = \sin y$

2.

Solve: $(D^2-DD'-20D'^2)z = e^{5x+y}+\sin(4x-y)$

3.

Solve: $(D^2+DD'-6D'^2)z = x^2y+e^{3x+y}$

4.

Solve: $(D^2-4DD'+4D'^2)z = xy+e^{2x+y}$

5.

Solve: $(D^2+4DD'-5D'^2)z = \sin(x-2y)+e^{2x-y}$

6.

Solve: $(4D^2-4DD'+D'^2)z = 16\log(x+2y)$