CHAPTER 1
VECTOR ANALYSIS

1.0 Objective: To learn mathematical as well as graphical presentation of a vector and its unit vector.

1.1 Unit Vector: A vector '**A**' is represented by its magnitude$|\mathbf{A}| = \mathrm{A}$ and a unit vector $\mathbf{a_A}$ indicating the direction of **A** along **A.** Hence, $\mathbf{a_A} = \frac{\mathbf{A}}{|\mathbf{A}|}$ and $|\mathrm{a_A}| = 1$. Thus **A** may be written as $\mathbf{A} = \mathrm{A}\mathbf{a_A}$**.**

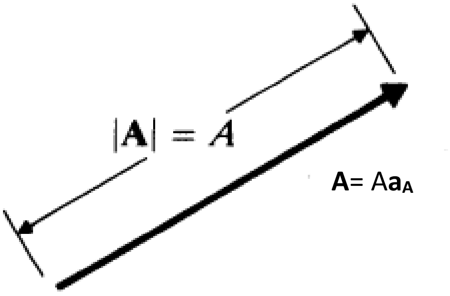
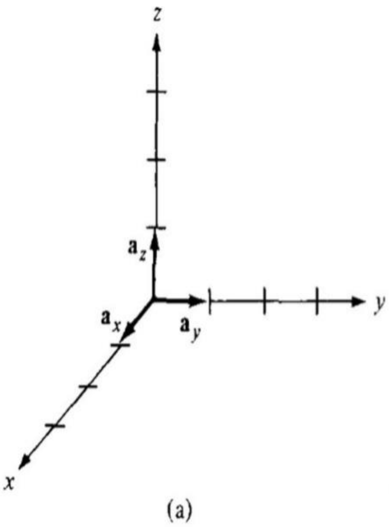
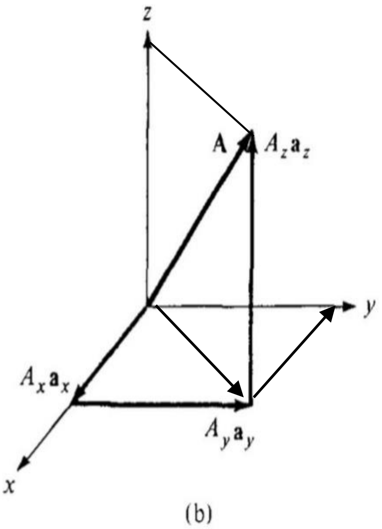


Figure: 1(a) Graphical representation of vector **A**





Vector $\mathbf{A}=A_x\mathbf{a_x}+A_y\mathbf{a_y}+A_z\mathbf{a_z}$

$A\rho 2= A_x^2+A_y^2$

Magnitude $A=\sqrt{(A\rho 2 + A_z^2)}$

Magnitude $A=\sqrt{(A_x^2+A_y^2+A_z^2)}$

$$\mathbf{a_A} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

Figure 1 (b): Unit vectors a_x, a_y and a_z Figure 1 (c): Components of A along a_x, a_y and a_z

Problem: 1.1

Find the unit vector of $\mathbf{A}=5\mathbf{a_x}-2\mathbf{a_y}+\mathbf{a_z}$

Solution:

$|\mathbf{A}|=\sqrt{(A_x^2+A_y^2+A_z^2)}$

$= \sqrt{(5^2+2^2+1^2)}= \sqrt{30}=5.48$

$$\boldsymbol{\alpha}_\mathbf{A}= \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{5\mathbf{ax}-2\mathbf{ay}+\mathbf{az}}{5.48} = 0.912\mathbf{a_x}- 0.365\mathbf{a_y}+0.1825\mathbf{a_z}$$

1.2 Location of a point in space in different coordinates

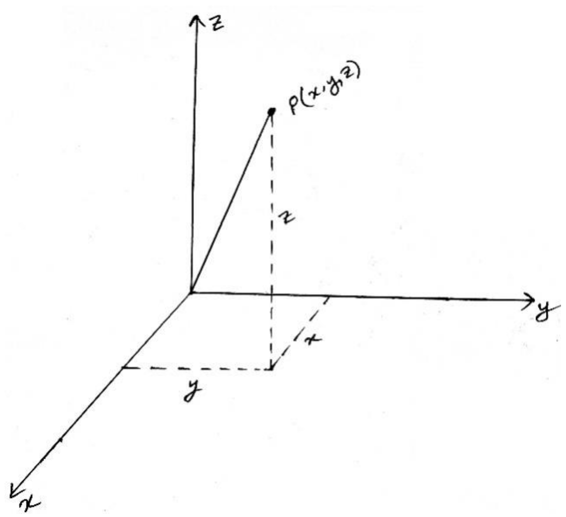


Figure: 1(d) P(x, y, z) in Rectangular form

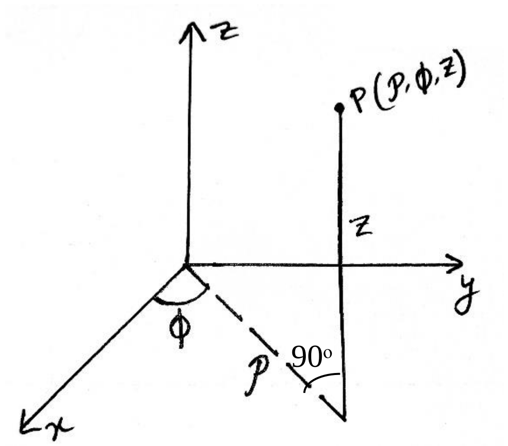


Figure: 1(e) P (ρ, φ, z) in Cylindrical form

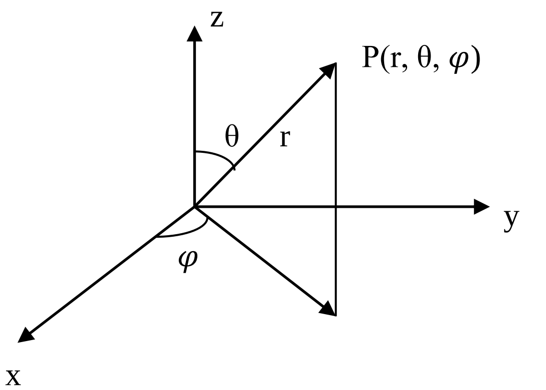
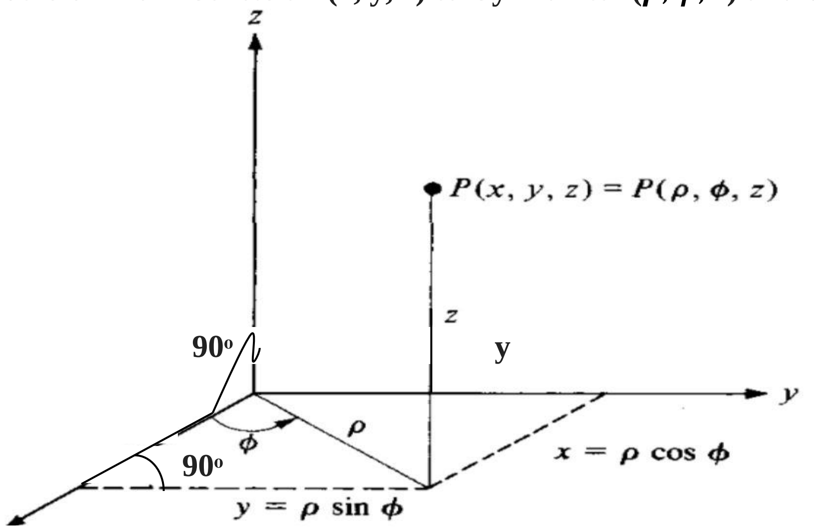


Figure: 1(f) P (r, θ, φ) in Spherical form

1.3.1 Coordinate conversion from Cartesian (x, y, z) to Cylindrical (ρ, φ, z) and vice versa



$\frac{y}{\rho} = \sin\varphi$ $\frac{x}{\rho} = \cos\varphi$

Figure: 2 (a) Coordinate conversion from Cartesian to Cylindrical and vice versa.

Cartesian to Cylindrical-

$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$

$\tan\varphi = \frac{y}{x}$

$\varphi = \tan^{-1}\frac{y}{x}$

z = z

Cylindrical to Cartesian-

x = $\rho \; cos \; \varphi$

y = $\rho \; sin \; \varphi$

z = z

1.3 .2 Coordinate conversion from Cartesian (x, y, z), to Spherical ($\boldsymbol{r}, \boldsymbol{\varphi}, \boldsymbol{\theta}$) and vice versa

$\sin\theta = \frac{\rho}{r}$

$\text{Cos}\theta = \frac{z}{r}$

$\rho^2 = x^2 + y^2$

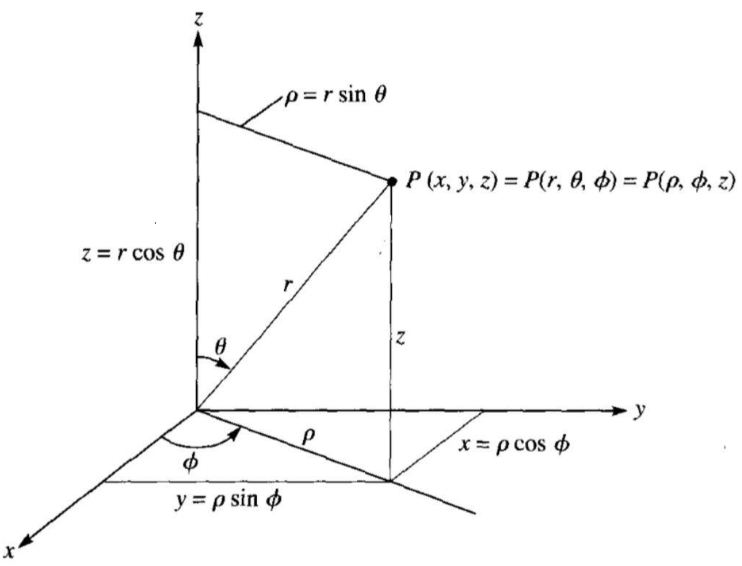


Figure: 2(b) Conversion of (x,y, z) to (r, $\boldsymbol{\theta}, \boldsymbol{\varphi}$) vice versa

Cartesian to Spherical-

$r = \sqrt{\rho^2 + z^2} = \sqrt{x^2 + y^2 + z^2} \quad (r \geq 0)$

$\varphi = \tan^{-1}\frac{y}{x}$

$\tan\theta = \frac{\rho}{z}$

$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \cos^{-1}\frac{z}{r} \quad (0° \leq \theta \leq 180°)$

Spherical to Cartesian-

x = ρ cos φ

x = $r \; sin\theta \; cos \; \varphi$

y = ρ sin φ

y = $r \; sin\theta \; sin \; \varphi$

z = $r \; cos\theta$

Vector presented in different coordinate system

$\mathbf{A} = A_x\mathbf{a_x} + A_y\mathbf{a_y} + A_z\mathbf{a_z}$ (Rectangular)

$= A_\rho\mathbf{a_\rho} + A\varphi\mathbf{a_\theta} + A_z\mathbf{a_z}$ (Cylindrical)

$= A_r\mathbf{a_r} + A_\theta\mathbf{a_\theta} + A\varphi\mathbf{a}\boldsymbol{\varphi}$ (Spherical)

Problem-1.2:

If P (3, 4, 5). Evaluate P (ρ, φ, z) and P $(r, \theta, \varphi,)$.

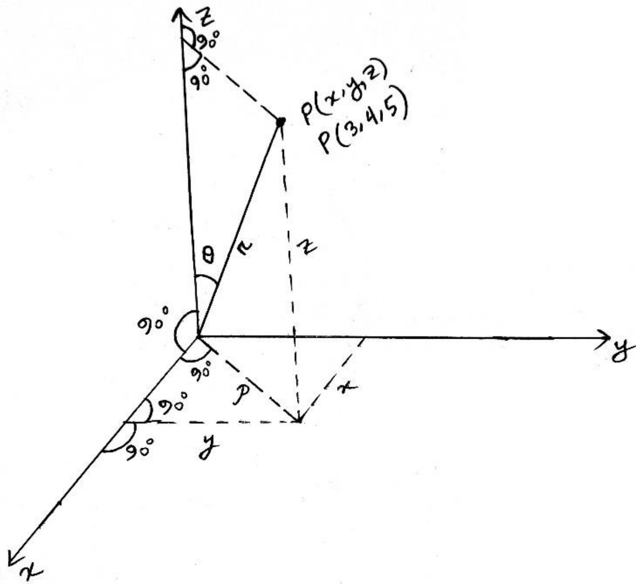


Figure: 3

Solution:

Given, x=3, y=4, & z=5

For Cylindrical Co-ordinate,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$$

$$\tan\varphi = \frac{y}{x} \qquad \therefore \varphi = \tan^{-1}\frac{4}{3} = 53.13^0 \quad \text{And } z=5$$

$$\therefore \ P(\rho, \varphi, z) = P(5, 53.13^0, \ 5)$$

For Spherical Co-ordinate,

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + 4^2 + 5^2} \quad = 7.07,$$

$$\text{Cos } \theta = \frac{z}{r}$$

$$\theta = \cos^{-1}\frac{5}{7.07} \quad = 45^0$$

$$\tan\varphi = \frac{y}{x}; \qquad \varphi = \tan^{-1}\frac{4}{3}$$

$$\text{And } \ \varphi = 53.13^0$$

$$\therefore \ P(r, \theta, \varphi) = P(7.07, 45^0, \ 53.13^0)$$

Problem 1.3: The position of a point P(8, 120°, 330°) specify its location in terms of (a) Rectangular and (b) Cylindrical coordinate (c) Position vector OP for each case (vector going from origin to P).

Given, r =8, $\theta = 120°$ and $\varphi = 330°$.

(a) find the location of P (x,y,z)

x = r$sin\theta\ cos\varphi$= 8sin(120°)cos(330°) = 6 ;

y = r sin(θ)sin(φ) = 8sin(120°)sin(330°)= - 2√3 and

z = rcosθ =8 cos(120°)= - 4

Hence location of P (x,y,z) = (6, -2√3, -4)

(b) Find the location of P(ρ, φ, z)

ρ= r$sin\theta$ = 8sin(120°) = 6.92,

$\varphi = 330°$ and

Z = rco$s\theta$ = 8cos (120°) = -4

Hence P (ρ, φ, z) = P (6.92, 330°, -4)

(c)

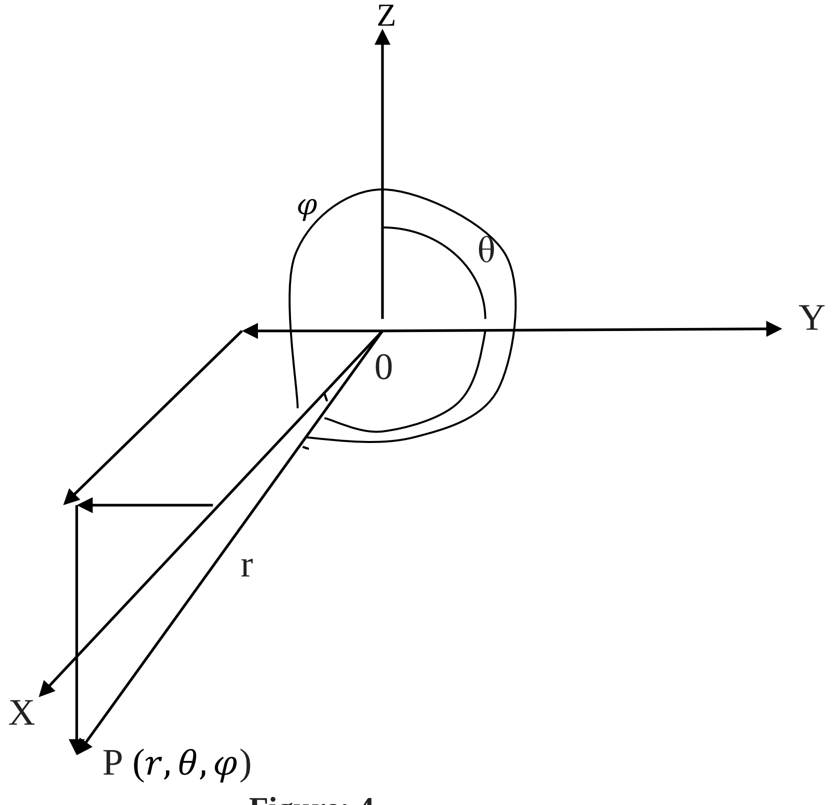


Figure: 4

OP= x$\mathbf{a_x}$ + y$\mathbf{a_y}$+z$\mathbf{a_z}$

= 6$\mathbf{a_x}$ -2√3$\mathbf{a_y}$ -4$\mathbf{a_z}$ (Rectangular)

OP= ρ$\mathbf{a_\rho}$+ z$\mathbf{a_z}$

= 6.92$\mathbf{a_\rho}$ – 4$\mathbf{a_z}$ (Cylindrical)

OP= r$\mathbf{a_r}$

= 8$\mathbf{a_r}$ (Spherical)

1.0 (2) Objective: To learn vector summation subtraction and product.

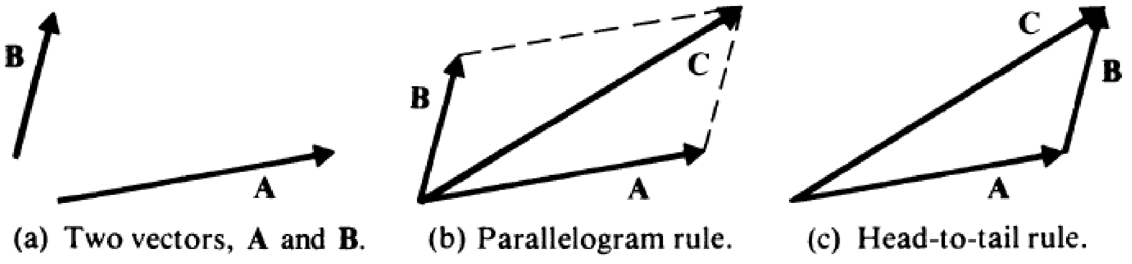


Figure 5(a): Vector addition

Vector addition, **C** = **A+ B**

Problem 1.4: Show the equation of (a) Addition of the following two vectors and the Subtraction of two vectors

A= $2\boldsymbol{a}_x - 0\boldsymbol{a}_y + \boldsymbol{a}_z$ and **B**= $0\boldsymbol{a}_x + 2\boldsymbol{a}_y - 4\boldsymbol{a}_z$

Solution:

1.4 (a) Vector Addition $\mathbf{C}_1$ = **A+B** = **(**$2\boldsymbol{a}_x - 0\boldsymbol{a}_y + \boldsymbol{a}_z$) + **(** $0\boldsymbol{a}_x + 2\boldsymbol{a}_y - 4\boldsymbol{a}_z$**)**

$= 2\boldsymbol{a}_x + 2\boldsymbol{a}_y - 3\boldsymbol{a}_z$

1.4(b) Vector Subtraction $\mathbf{C}_2$ = **A − B** = **(**$2\boldsymbol{a}_x - 0\boldsymbol{a}_y + \boldsymbol{a}_z$**)** – ($0\boldsymbol{a}_x + 2\boldsymbol{a}_y - 4\boldsymbol{a}_z$)

$= (2\boldsymbol{a}_x - 0\boldsymbol{a}_y + \boldsymbol{a}_z$ - $0\boldsymbol{a}_x - 2\boldsymbol{a}_y + 4\boldsymbol{a}_z)$

$= 2\boldsymbol{a}_x - 2\boldsymbol{a}_y + 5\boldsymbol{a}_z$

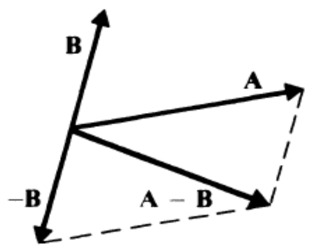


Figure 5(b): Vector subtraction

Dot product and cross product

A.B = ABcos θ

= ABcos0° =AB(1)

= AB cos 90° = 0

$\boldsymbol{a}_x.\boldsymbol{a}_x = cos0° = 1$

$\boldsymbol{a}_x.\boldsymbol{a}_y = cos90° = 0$

$\boldsymbol{a}_x.\boldsymbol{a}_z = 0$

$\boldsymbol{a}_y.\boldsymbol{a}_y = 1$

$\boldsymbol{a}_y.\boldsymbol{a}_z = 0$

$\boldsymbol{a}_z.\boldsymbol{a}_z = 1$

A× $\boldsymbol{B} = ABsin\theta$**a**$_n$

= AB sin0° =0

= ABsin90°=AB (1)

$\boldsymbol{a}_x \times \boldsymbol{a}_x = sin0° = 0$

$\boldsymbol{a}_x \times \boldsymbol{a}_y = sin90° a_z = \boldsymbol{a}_z$

$\boldsymbol{a}_x \times \boldsymbol{a}_z = \boldsymbol{a}_y$

$\boldsymbol{a}_y \times \boldsymbol{a}_y = 0$

$\boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$

$\boldsymbol{a}_z \times \boldsymbol{a}_z$ =0

1.4(c) Dot product of vectors **A.B** = ABcosθ =AxBx + AyBy + AzBz (Rectangular)

A= Ax$\boldsymbol{a_x} + Ay\boldsymbol{a_y} + Az\boldsymbol{a_z}$ $and\ B = Bx\boldsymbol{b_x} + By\boldsymbol{b_y} + Bz\boldsymbol{b_z}$

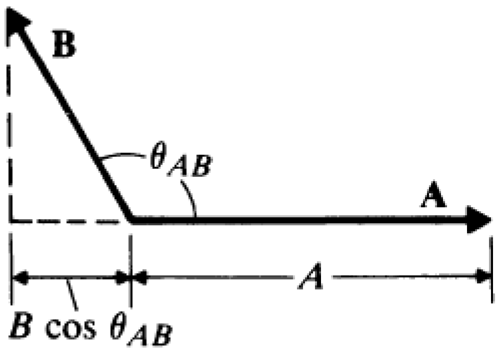


Figure 5(c): Dot product of **A** and **B**

C_3 = A.B = ($2\boldsymbol{a_x} + \boldsymbol{a_z}$). ($2\boldsymbol{a_y} - 4\boldsymbol{a_z}$)

$= (2\boldsymbol{a_x}. 2\boldsymbol{a_y} + \boldsymbol{a_z}. 2\boldsymbol{a_y} + 2\boldsymbol{a_x}. (-4\boldsymbol{a_z}) + \boldsymbol{a_z}. (-4\boldsymbol{a_z}))$

= 4 (0) + 2(0) + 8(0) - 4(1)

= 0−4 = −4

A= Ax$\boldsymbol{a_x} + Ay\boldsymbol{a_y} + Az\boldsymbol{a_z}$ $and\ \boldsymbol{B} = Bx\boldsymbol{b_x} + By\boldsymbol{b_y} + Bz\boldsymbol{b_z}$

A.B = Ax$Bx + AyBy + AzBz$

1.4(d) Cross product **C_4** = **A**$\times$ $\boldsymbol{B}$ = $\begin{bmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ 2 & -0 & 1 \\ 0 & 2 & -4 \end{bmatrix}$

$=[(-0)(-4) - (1)(2)]\ \boldsymbol{a_x} - [(2)(-4) - (1)(0)]\boldsymbol{a_y} + [(2)(2) - (0)(-0)]\boldsymbol{a_z}$

$= -2\boldsymbol{a_x} - 8\boldsymbol{a_y} + 4\boldsymbol{a_z}$

A$\times$ $\boldsymbol{B}$ = $\begin{bmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \text{Ax} & \text{Ay} & Az \\ Bx & By & Bz \end{bmatrix}$= (Ay $Bz -\ Az\ By)\boldsymbol{a_x}$ − (Ax $Bz -\ AzBx)\ \boldsymbol{a_y}$+ **(**Ax $By -$ Ay $Bx)\ \boldsymbol{a_z}$

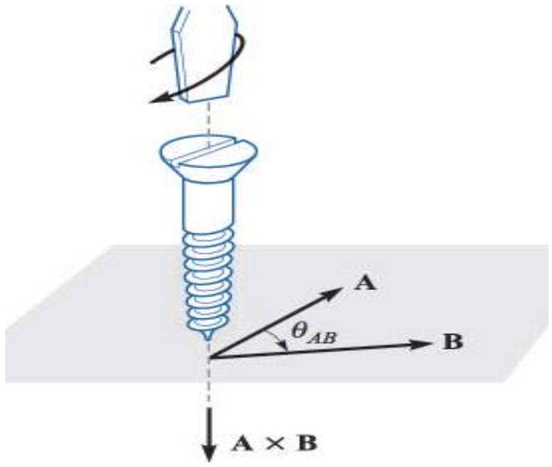


Figure 5(d): Cross product of A and B, (**A**x**B** =ABsinθa_n)

Problem -1.5:

Write the vector equation of line (P_1P_2) joining point P_1 (1, 3, 2) and point P_2 (3,-2, 4). What is the length of the Line?

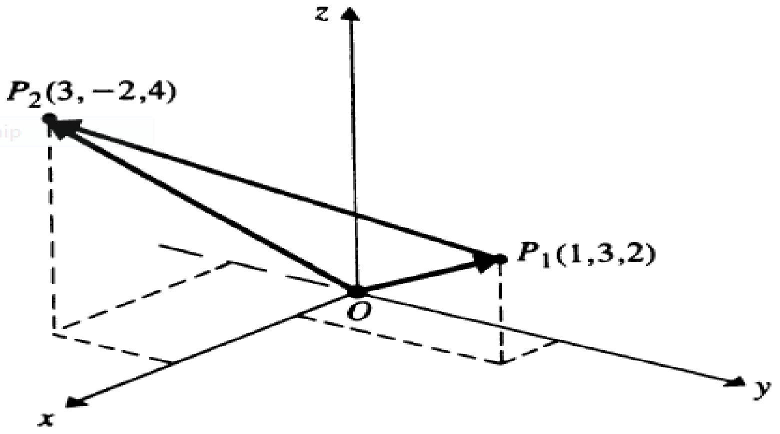


Figure: 6

From figure 6

$$\mathbf{P_1P_2} = \mathbf{OP_2} - \mathbf{OP_1}$$

$$= [(3\text{-}0)\,\mathbf{a_x} + (-2\text{-}0)\mathbf{a_y} + (4\text{-}0)\mathbf{a_z}] - [(1\text{-}0)\mathbf{a_x} + (3\text{-}0)\mathbf{a_y} + (2\text{-}0)\mathbf{a_z})]$$

$$= 2\mathbf{a_x} - 5\mathbf{a_y} + 2\mathbf{a_z}$$

Length of line:

$$P_1P_2 = |P1P2|$$

$$= \sqrt{[(2)^2 + (-5)^2 + (2)^2]}$$

$$= \sqrt{33}$$

1.0(3) Objective: To learn mathematical and graphical presentation of differential volume, surface, and length in different coordinate system.

1.5 Rectangular co-ordinate system:

A differential volume element in the rectangular coordinate system is developed by incremental differential changes dx, dy, and dz along the unit vectors a_x, a_y and a_z, respectively, as illustrated in following Figure 7.

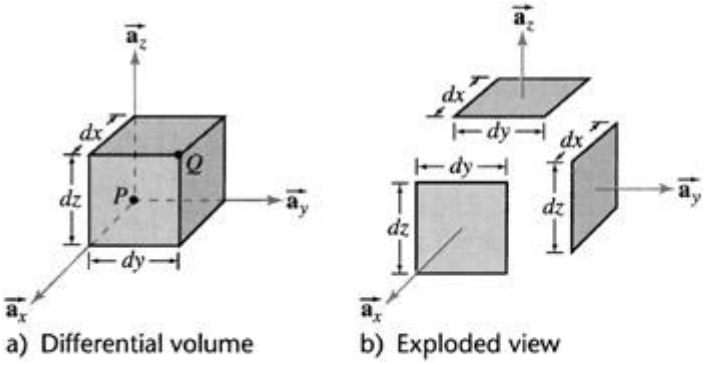


Figure: 7 Differential elements in rectangular coordinate

The volume is enclosed by six differential surfaces. Each surface is defined by a unit vector normal to that surface. The following equations of differential lengths surfaces are:

The differential surfaces:

$d\mathbf{s}_{front} = dy\mathbf{a_y} \times dz\mathbf{a_z} = dydz\ \mathbf{a_x}$

$d\mathbf{s}_{back} = dydz\ (-\mathbf{a_x})$

$d\mathbf{s}_{right} = dx\mathbf{a_x} \times dz\mathbf{a_z} = dx\ dz\ \mathbf{a_y}$

$d\mathbf{s}_{left} = dx\ dz\ (-\mathbf{a_y})$

$d\mathbf{s}_{top} = dx\mathbf{a_x} \times dy\mathbf{a_y} = dx\ dy\ \mathbf{a_z}$

$d\mathbf{s}_{bottom} = dx\ dy\ (-\mathbf{a_z})$

The differential lengths:

$d\mathbf{l_x} = dx\mathbf{a_x}$

$d\mathbf{l_y} = dy\mathbf{a_y}$

$d\mathbf{l_z} = dz\mathbf{a_z}$

The general equation of differential length:

$\mathbf{dl} = dl_x\mathbf{a_x} + dl_y\mathbf{a_y} + dl_z\mathbf{a_z}$

The equation of differential volume is:

$dv = d\mathbf{s}.d\mathbf{l} = d\mathbf{s}_f.d\mathbf{l}_x$

$= dydz\mathbf{a_x}.dx\mathbf{a_x}$

$dv = dxdydz$

Problem-1.6:

A rectangular box having length**,** width and height are 4,3,2 meters. Evaluate the volume, top and back surfaces and diagonal of the volume.

Solution:

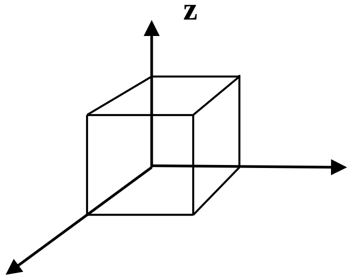


Figure: 8

Given x=4, y=3, and z=2

The Equation of differential volume dv= dxdydz

$$\int dv = \int_0^2 \int_0^3 \int_0^4 dx\, dy\, dz$$

v= $[x]_0^4\,[y]_0^3[z]_0^2$

v= [4-0][3-0][2-0] = $24m^3$

The equation of differential top surface $d\mathbf{s}_{top}$ = dx dy $\mathbf{a_z}$

$\int_s \; d\mathbf{s}_{top} = \int_0^3 \int_0^4 dx\, dy\, \boldsymbol{a_z}$

$\mathbf{S}_{top}$ = $[x]_0^4[y]_0^3\boldsymbol{a_z}$

$\mathbf{S}_{top}$ = [4-0][3-0]$\boldsymbol{a_z}$ = $12\boldsymbol{a_z}m^2$

$d\mathbf{s}_{back}$ = dydz $(-\mathbf{a_x})$

$\int d\mathbf{s}_{back} = \int\int dydz(-\boldsymbol{a_x})$

$\mathbf{s}_{back} = -\int_0^2 \int_0^3 dydz\, \boldsymbol{a_x}$

$\mathbf{s}_{back} = -\,[y]_0^3[z]_0^2\, \boldsymbol{a_x} = -6\, \boldsymbol{a_x}m^2$

The diagonal length of the differential volume:

$\mathbf{l} = l_x\mathbf{a_x} + l_y\mathbf{a_y} + l_z\mathbf{a_z}$ $\quad\quad =\gg d\mathbf{l} = dl_x\mathbf{a_x} + dl_y\mathbf{a_y} + dl_z\mathbf{a_z}$

$\int \mathbf{dl} = \int_0^4 dx\mathbf{a_x} + \int_0^3 dy\mathbf{a_y} + \int_0^2 dz\mathbf{a_z}$

$\mathbf{l} = [x\mathbf{a_x}]_0^4 + [y\mathbf{a_y}]_0^3 + [z\mathbf{a_z}]_0^2$

$\mathbf{l} = 4\mathbf{a_x} + 3\mathbf{a_y} + 2\mathbf{a_z}$

1.6 Cylindrical co-ordinate system:

The differential volume elements:

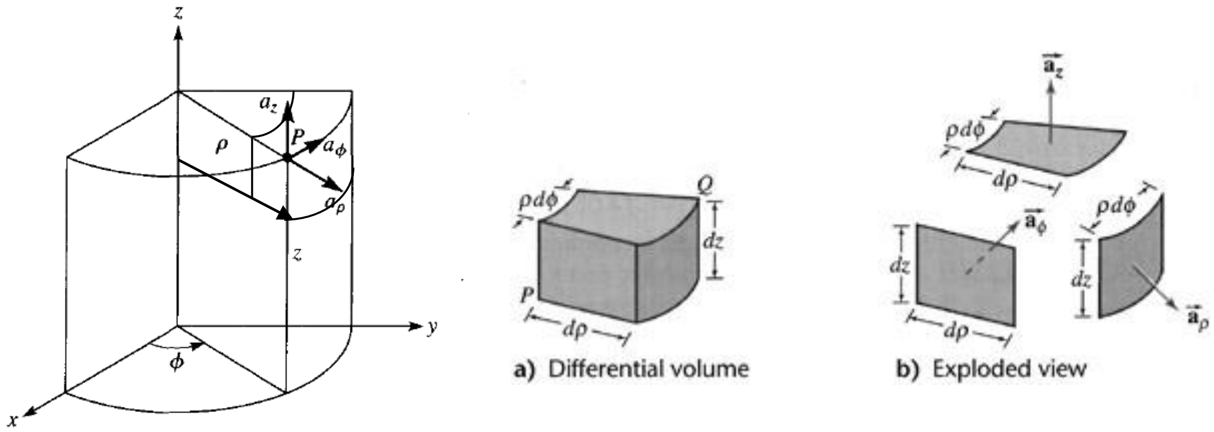


Figure 9: Differential volume elements

The differential surfaces:

$d\mathbf{s}_{front} = \rho d\varphi \mathbf{a}_{\boldsymbol{\varphi}} \times dz\mathbf{a}_{\mathbf{z}}$

$= \rho d\varphi dz\mathbf{a}_{\boldsymbol{\rho}}$

$d\mathbf{s}_{\mathbf{side}} = d\rho \mathbf{a}_{\boldsymbol{\rho}} \times dz\mathbf{a}_{\mathbf{z}}$

$= d\rho dz\ \mathbf{a}_{\boldsymbol{\varphi}}$

$d\mathbf{s}_{\mathbf{top}} = d\rho \mathbf{a}_{\boldsymbol{\rho}} \times \rho d\varphi \mathbf{a}_{\boldsymbol{\varphi}}$

$= \rho d\rho d\varphi\ \mathbf{a}_{\mathbf{z}}$

The equation of differential lengths:

$d\mathbf{l}_{\boldsymbol{\rho}} = d\rho \mathbf{a}_{\boldsymbol{\rho}}$

$d\mathbf{l}\boldsymbol{\varphi} = \rho d\varphi \mathbf{a}_{\boldsymbol{\varphi}}$

$d\mathbf{l}\mathbf{z} = dz\mathbf{a}_{\mathbf{z}}$

The differential length in Cylindrical coordinates:

$\bar{\mathbf{l}} = l_\rho \mathbf{a}_{\boldsymbol{\rho}} + l\varphi \mathbf{a}\boldsymbol{\varphi} + l_z \mathbf{a}_{\mathbf{z}}$ $\Rightarrow$ $\overline{d\mathbf{l}} = dl_\rho \mathbf{a}_{\boldsymbol{\rho}} + dl\varphi \mathbf{a}\boldsymbol{\varphi} + dl_z \mathbf{a}_{\mathbf{z}}$

$\overline{d\mathbf{l}} = d\rho \mathbf{a}_{\boldsymbol{\rho}} + \rho d\varphi \mathbf{a}_{\boldsymbol{\varphi}} + dz\mathbf{a}_{\mathbf{z}}$

The differential volume:

$dv = d\mathbf{s}_{\mathbf{f}} . d\mathbf{l}\boldsymbol{\rho}$

$= \rho d\varphi dz\ \mathbf{a}_{\boldsymbol{\rho}} . d\rho \mathbf{a}_{\boldsymbol{\rho}}$

$dv = \rho d\rho d\varphi dz$

Problem-1.7:

The height of a Cylinder is *h*, and its radius is r. Evaluate the volume and the side surface of the cylinder.

Solution:

The equation of the differential volume dv= $\rho\ d\rho\ d\varphi\ dz$

$$\text{The volume, } \int dv = \int_0^r \int_0^{2\pi} \int_0^h \rho\ d\rho\ d\varphi\ dz$$

$$v= \left[\frac{\rho^2}{2}\right]_0^r [\varphi]_0^{2\pi} [z]_0^h$$

$$v = \frac{1}{2} r^2 (2\pi)h$$

$$v= \pi r^2 h$$

The equation of the differential side surface $d\mathbf{s_s} = \rho d\varphi dz \mathbf{a_\rho}$

The side surface,

$$\oint d\mathbf{s_s} = \int_0^{2\pi} \int_0^h \rho d\varphi dz \mathbf{a_\rho}$$

$$\mathbf{s_s} = r(2\pi)h\mathbf{a_p}$$

$$\mathbf{s_s} = 2\pi r h\mathbf{a_p}$$

1.7 Spherical co-ordinate system:

The differential volume elements:

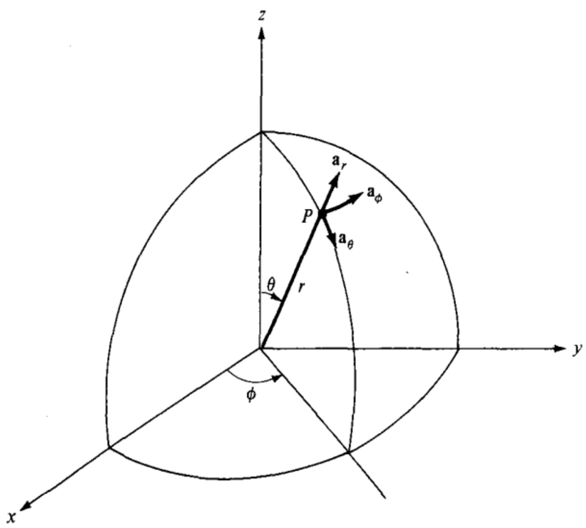


Figure: 10(a) Point **P** and unit vectors in Spherical coordinates

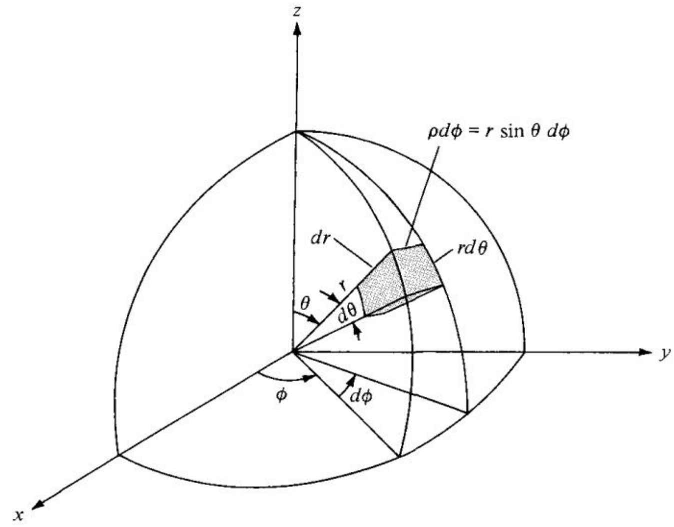


Figure: 10(b) differential elements in the spherical coordinate system

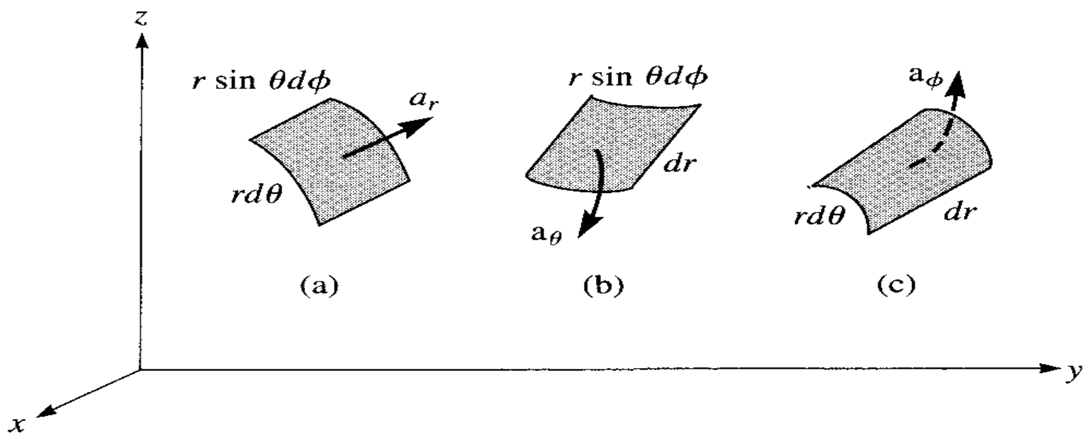


Figure :11(c) Differential normal areas in spherical coordinates

The differential surfaces

$d\mathbf{s}_{front} = (rd\Theta)\mathbf{a}_\theta \times (rSin\Theta d\varphi)\boldsymbol{a}_\varphi = r^2 Sin\Theta d\Theta d\varphi \mathbf{a}_r$

$d\mathbf{s}_{side} = (dr)\mathbf{a}_r \times (rd\Theta)\mathbf{a}_\theta = rdrd\Theta\mathbf{a}_\varphi$

$d\mathbf{s}_{top} = rSin\Theta d\varphi\boldsymbol{a}_\varphi \times dr\mathbf{a}_r = rSin\Theta drd\varphi\mathbf{a}_\theta$

The differential length

$\bar{\mathbf{l}} = l_r\mathbf{a}_r + l_\theta\mathbf{a}_\theta + l\varphi\mathbf{a}_\varphi \quad =\gg \quad \bar{d\mathbf{l}} = dl_r\mathbf{a}_r + dl_\theta\mathbf{a}_\theta + dl\varphi\mathbf{a}_\varphi$

$\bar{d\mathbf{l}} = dr\mathbf{a}_r + rd\Theta\mathbf{a}_\theta + r\, Sin\Theta\, d\varphi\mathbf{a}_\varphi$

The differential volume

$dv = d\mathbf{s}_{front} \cdot dr\mathbf{a_r}$

$= r^2 Sin\Theta d\varphi d\Theta \mathbf{a_r} \cdot dr\mathbf{a_r}$

$dv = r^2 Sin\Theta dr d\Theta d\varphi$

Problem-1.8:

The radius is 5m of a sphere. Evaluate the volume and surface of the sphere.

Solution:

The differential volume $dv = r^2 Sin\Theta dr d\Theta d\varphi$

$$\int dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{5} r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$v = \left[\frac{r^3}{3}\right]_0^5 [-\cos\theta]_0^{\pi} [\varphi]_0^{2\pi}$$

$$v = \frac{1}{3} 5^3 (2)(2\pi)$$

$$v = \frac{4}{3}\pi 5^3 \ =523.599 \ m^3$$

The differential surface $d\mathbf{s}_{front} = r^2 Sin\Theta d\varphi \mathbf{a_r}$

Surface, $\int d\mathbf{s} = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \, d\theta \, d\varphi \, \mathbf{a_r}$

$$\mathbf{s} = {r_1}^2 [-\cos\theta]_0^{\pi} [\varphi]_0^{2\pi} \mathbf{a_r}$$

$$\mathbf{s} = {r_1}^2 [-(\cos\pi - \cos 0)][2\pi - 0]\mathbf{a_r}$$

$$\mathbf{s} = 4\pi {r_1}^2 \mathbf{a_r} \quad \text{(Ans)}$$

1.8 Dot and cross product of unit vectors:

$$\mathbf{a_x}.\mathbf{a_x} = \cos 0^o = 1 \qquad \mathbf{a_x} \times \mathbf{a_x} = \sin 0^o = 0$$
$$\mathbf{a_x}.\mathbf{a_y} = \cos 90^o = 0 \qquad \mathbf{a_x} \times \mathbf{a_y} = \sin 90^o \mathrm{a_z} = \mathbf{a_z}$$
$$\mathbf{a_x}.\mathbf{a_z} = 0 \qquad \mathbf{a_x} \times \mathbf{a_z} = \mathbf{a_y}$$
$$\mathbf{a_y}.\mathbf{a_y} = 1 \qquad \mathbf{a_y} \times \mathbf{a_y} = 0$$
$$\mathbf{a_y}.\mathbf{a_z} = 0 \qquad \mathbf{a_y} \times \mathbf{a_z} = \mathbf{a_x}$$
$$\mathbf{a_z}.\mathbf{a_z} = 1 \qquad \mathbf{a_z} \times \mathbf{a_z} = 0$$

A.B = |**A**||**B**|cosθ

$\mathbf{a_x}.\mathbf{a_\rho} = (1)(1)\cos\varphi = \cos\varphi$

$\mathbf{a_y}.\mathbf{a_\rho} = (1)(1)\cos(90 - \varphi) = \sin\varphi$

$\mathbf{a_x}.\mathbf{a_\varphi} = (1)(1)\cos(90 + \varphi) = -\sin\varphi$

$\mathbf{a_y}.\mathbf{a_\varphi} = (1)(1)\cos(\varphi) = \cos\varphi$

$\mathbf{a_x}.\mathbf{a_z} = \mathbf{a_y}.\mathbf{a_z} =$

$\mathbf{a_z}.\mathbf{a_\rho} = \mathbf{a_z}.\mathbf{a_\varphi} = (1)(1)\cos(90^\circ) = 0$

$\mathbf{a_z}.\mathbf{a_z} = (1)(1)\cos(0^\circ) = 1$

1.9 (a) Dot products of unit vectors (Cartesian and Cylindrical)

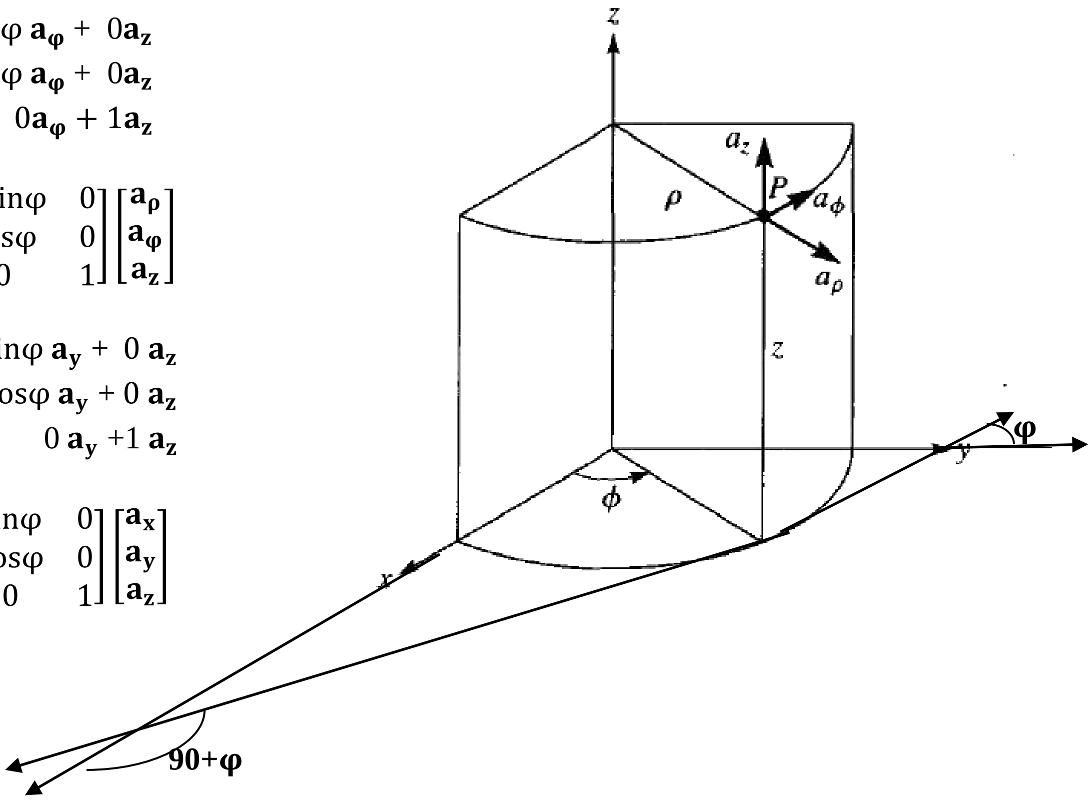
	$\mathbf{a_\rho}$	$\mathbf{a_\varphi}$	$\mathbf{a_z}$
$\mathbf{a_x}.$	cosφ	-sinφ	0
$\mathbf{a_y}.$	sinφ	cosφ	0
$\mathbf{a_z}.$	0	0	1

$$\mathbf{a_x} = \cos\varphi\, \mathbf{a_\rho} - \sin\varphi\, \mathbf{a_\varphi} + 0\mathbf{a_z}$$
$$\mathbf{a_y} = \sin\varphi\, \mathbf{a_\rho} + \cos\varphi\, \mathbf{a_\varphi} + 0\mathbf{a_z}$$
$$\mathbf{a_z} = 0\, \mathbf{a_\rho} + 0\mathbf{a_\varphi} + 1\mathbf{a_z}$$

$$\begin{bmatrix}\mathbf{a_x}\\ \mathbf{a_y}\\ \mathbf{a_z}\end{bmatrix} = \begin{bmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{a_\rho}\\ \mathbf{a_\varphi}\\ \mathbf{a_z}\end{bmatrix}$$

$$\mathbf{a_\rho} = \cos\varphi\, \mathbf{a_x} + \sin\varphi\, \mathbf{a_y} + 0\, \mathbf{a_z}$$
$$\mathbf{a_\varphi} = -\sin\varphi\, \mathbf{a_x} + \cos\varphi\, \mathbf{a_y} + 0\, \mathbf{a_z}$$
$$\mathbf{a_z} = 0\, \mathbf{a_x} + 0\, \mathbf{a_y} + 1\, \mathbf{a_z}$$

$$\begin{bmatrix}\mathbf{a_\rho}\\ \mathbf{a_\varphi}\\ \mathbf{a_z}\end{bmatrix} = \begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{a_x}\\ \mathbf{a_y}\\ \mathbf{a_z}\end{bmatrix}$$



$\mathbf{a_x}.\mathbf{a_\varphi} = -\sin\varphi$

$\mathbf{a_x}.\mathbf{a_z} = 0$

$\mathbf{a_x}.\mathbf{a_\rho} = \cos\varphi$

ay.aρ=sinφ

$\mathbf{a_y}.\mathbf{a_\varphi} = \cos\varphi$

$\mathbf{a_y}.\mathbf{a_z} = 0$

$\mathbf{a_z}.\mathbf{a_\rho} = 0$

$\mathbf{a_z}.\mathbf{a_\varphi} = 0$

$\mathbf{a_z}.\mathbf{a_z} = 1$

Problem: 1.9(a)

Convert the vector F= $xy\mathbf{a_x} - 2x\mathbf{a_y}$ **into the Cylindrical Coordinate system**

$$\begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} => \begin{bmatrix} F_\rho \\ F_\varphi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xy \\ -2x \\ 0 \end{bmatrix}$$

$F_\rho = xy\cos\varphi - 2x\sin\varphi$

$F_\varphi = -xy\sin\varphi - 2x\cos\varphi$,

$F_z = 0$

F $= F_\rho\mathbf{a_\rho} + F_\varphi\mathbf{a_\varphi} + F_Z\mathbf{a_z}$

$= (xy\cos\varphi - 2x\sin\varphi)\mathbf{a_\rho} - (xy\sin\varphi + 2x\cos\varphi)\mathbf{a_\varphi}$

$= (\rho\cos\varphi\rho\sin\varphi\cos\varphi - 2\rho\cos\varphi\sin\varphi)\mathbf{a_\rho} - (\rho\cos\varphi\sin\varphi\sin\varphi + 2\rho\cos\varphi\cos\varphi)\mathbf{a_\varphi}$

F $= \rho\cos\varphi\sin\varphi(\rho\cos\varphi - 2)\mathbf{a_\rho} - \rho\cos\varphi(\sin^2\varphi + 2\cos\varphi)\mathbf{a_\varphi}$ (Ans.)

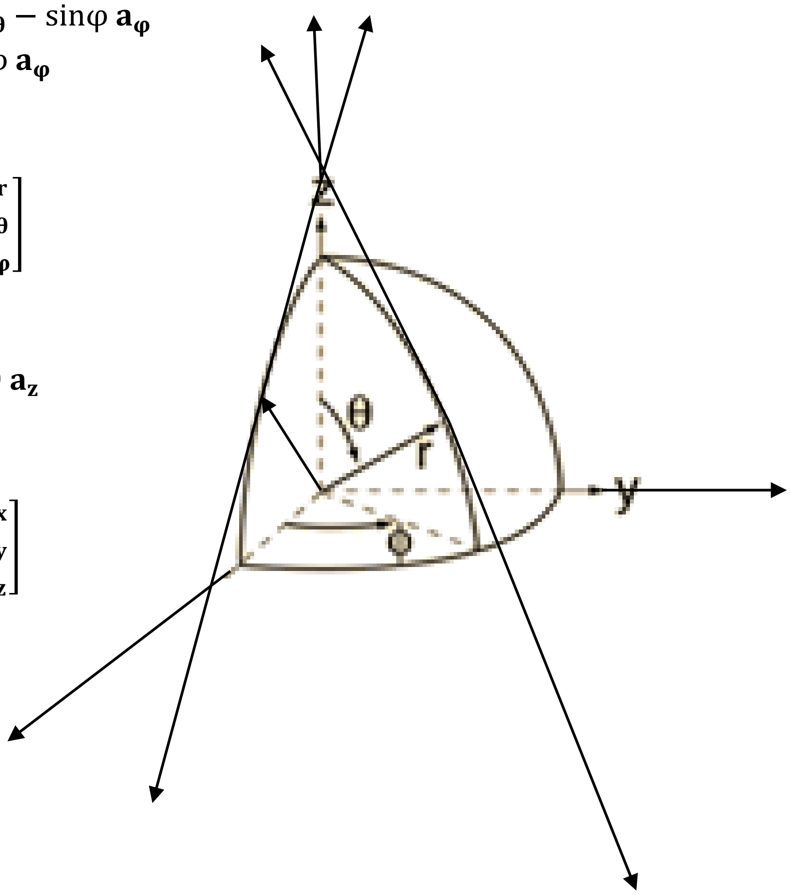
Assignment:

Convert B=$y\mathbf{a_x} + x\mathbf{a_y} + z\mathbf{a_z}$ **into (a) Cylindrical and (b) Spherical Coordinate system**

1.9 (b) Dot products of unit vectors (Cartesian and Spherical)

	$\mathbf{a_r}$	$\mathbf{a_\theta}$	$\mathbf{a_\varphi}$
$\mathbf{a_x}.$	$\sin\theta\cos\varphi$	$\cos\theta\cos\varphi$	$-\sin\varphi$
$\mathbf{a_y}.$	$\sin\theta\sin\varphi$	$\cos\theta\sin\varphi$	$\cos\varphi$
$\mathbf{a_z}.$	$\cos\theta$	$-\sin\theta$	0

$$\mathbf{a_x} = \sin\theta\cos\varphi\,\mathbf{a_r} + \cos\theta\cos\varphi\,\mathbf{a_\theta} - \sin\varphi\,\mathbf{a_\varphi}$$
$$\mathbf{a_y} = \sin\theta\sin\varphi\,\mathbf{a_r} + \cos\theta\sin\varphi\,\mathbf{a_\theta} + \cos\varphi\,\mathbf{a_\varphi}$$
$$\mathbf{a_z} = \cos\theta\,\mathbf{a_r} - \sin\theta\,\mathbf{a_\theta} + 0\mathbf{a_\rho}$$



$$\begin{bmatrix}\mathbf{a_x}\\ \mathbf{a_y}\\ \mathbf{a_z}\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi\\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi\\ \cos\theta & -\sin\theta & 0\end{bmatrix}\begin{bmatrix}\mathbf{a_r}\\ \mathbf{a_\theta}\\ \mathbf{a_\varphi}\end{bmatrix}$$

$$\mathbf{a_r} = \sin\theta\cos\varphi\,\mathbf{a_x} + \sin\theta\sin\varphi\,\mathbf{a_y} + \cos\theta\,\mathbf{a_z}$$
$$\mathbf{a_\theta} = \cos\theta\cos\varphi\,\mathbf{a_x} + \cos\theta\sin\varphi\,\mathbf{a_y} - \sin\theta\,\mathbf{a_z}$$
$$\mathbf{a_\varphi} = -\sin\varphi\,\mathbf{a_x} + \cos\varphi\,\mathbf{a_y} + 0\mathbf{a_z}$$

$$\begin{bmatrix}\mathbf{a_r}\\ \mathbf{a_\theta}\\ \mathbf{a_\varphi}\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta\\ -\sin\varphi & \cos\varphi & 0\end{bmatrix}\begin{bmatrix}\mathbf{a_x}\\ \mathbf{a_y}\\ \mathbf{a_z}\end{bmatrix}$$

$$\mathbf{a_x}.\mathbf{a_r} = \sin\theta\cos\varphi$$
$$\mathbf{a_x}.\mathbf{a_\theta} = \cos\theta\cos\varphi$$
$$\mathbf{a_x}.\mathbf{a_\varphi} = -\sin\varphi$$
$$\mathbf{a_y}.\mathbf{a_r} = \sin\theta\sin\varphi$$
$$\mathbf{a_y}.\mathbf{a_\theta} = \cos\theta\sin\varphi$$
$$\mathbf{a_y}.\mathbf{a_\varphi} = \cos\varphi$$
$$\mathbf{a_z}.\mathbf{a_r} = \cos\theta$$
$$\mathbf{a_z}.\mathbf{a_\theta} = -\sin\theta$$
$$\mathbf{a_z}.\mathbf{a_\varphi} = 0$$

A.B = $|\mathbf{A}||\mathbf{B}|\cos\theta$

$\mathbf{a_x}.\mathbf{a_r} = \mathbf{a_x}.\,\mathbf{a_\rho}\sin\theta = (1)(1)\cos\varphi\sin\theta = \sin\theta\cos\varphi$

$\mathbf{a_y}.\mathbf{a_r} = \mathbf{a_y}.\mathbf{a_\rho}\,\sin\theta = (1)(1)\cos(90-\varphi)\,\sin\theta = \sin\theta\sin\varphi$

$\mathbf{a_x}.\mathbf{a_\theta} = (1)(1)\cos\varphi\cos\theta = \cos\theta\cos\varphi$

$\mathbf{a_y}.\mathbf{a_\theta} = (1)(1)\sin\varphi\cos\theta = \cos\theta\sin\varphi$

$\mathbf{a_x}.\mathbf{a_\varphi} = (1)(1)\cos(90+\varphi) = -\sin\varphi$

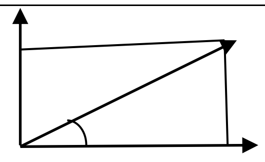
$\mathbf{a_y}.\mathbf{a_\varphi} = (1)(1)\cos(\varphi) = \cos\varphi$

$\mathbf{a_z}.\,\mathbf{a_r} = (1)(1)\cos\theta = \cos\theta$

$\mathbf{a_z}.\,\mathbf{a_\theta} = (1)(1)\cos(90+\theta) = -\sin\theta$

$\mathbf{a_z}.\,\mathbf{a_\varphi} = (1)(1)\cos(90) = 0$

$\mathbf{a_x}.\mathbf{a_z} = \mathbf{a_y}.\mathbf{a_z} = \mathbf{a_z}.\,\mathbf{a_\rho} = \mathbf{a_z}.\,\mathbf{a_\varphi} = \mathbf{a_z}.\,\mathbf{a_z} = (1)(1)\cos(90) = 0$



Problem: 1.10(a)

Convert **G** = $\frac{xz}{y}\boldsymbol{a_x}$ into Spherical Coordinate system

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} => \begin{bmatrix} G_r \\ G_\theta \\ G_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \frac{xz}{y} \\ 0 \\ 0 \end{bmatrix}$$

$G_r = \sin\theta\cos\varphi\,\frac{xz}{y} = \sin\theta\cos\varphi\,\frac{r\sin\theta\cos\varphi(r\cos\theta)}{r\sin\theta\sin\varphi}$

$G_\theta = \cos\theta\cos\varphi\,\frac{xz}{y} = \cos\theta\cos\varphi\,\frac{r\sin\theta\cos\varphi(r\cos\theta)}{r\sin\theta\sin\varphi}$

$G_\varphi = -\sin\varphi\,\frac{xz}{y} = -\sin\varphi\,\frac{r\sin\theta\cos\varphi(r\cos\theta)}{r\sin\theta\sin\varphi}$

G = $G_r\mathbf{a_r} + G_\theta\mathbf{a_\theta} + G_\varphi\mathbf{a_\varphi}$

$= \sin\theta\cos\varphi\,\frac{r\sin\theta\cos\varphi(r\cos\theta)}{r\sin\theta\sin\varphi}\,\mathbf{a_r} + \cos\theta\cos\varphi\,\frac{r\sin\theta\cos\varphi(r\cos\theta)}{r\sin\theta\sin\varphi}\,\mathbf{a_\theta} - \sin\varphi\,\frac{r\sin\theta\cos\varphi(r\cos\theta)}{r\sin\theta\sin\varphi}\,\mathbf{a_\varphi}$

$= r\cos\theta\cos\varphi\,(\sin\theta\cot\varphi\,\mathbf{a_r} + \cos\theta\cot\varphi\,\mathbf{a_\theta} - \mathbf{a_\varphi})$ (Ans.)

Alternate Method:

Problem-1.9 (b):

Convert the vector $B = y\mathbf{a_x} - x\mathbf{a_y} + z\mathbf{a_z}$ into cylindrical co-ordinates.

Solution:

The new components are

$$B_\rho = B\cdot\mathbf{a_\rho} = y(\mathbf{a_x}\cdot\mathbf{a_\rho}) - x(\mathbf{a_y}\cdot\mathbf{a\rho})$$

$$= y\cos\varphi - x\sin\varphi = \rho\sin\varphi\cos\varphi - \rho\cos\varphi\sin\varphi = 0$$

$$B_\varphi = B\cdot\mathbf{a_\varphi} = y(\mathbf{a_x}\cdot\mathbf{a_\varphi}) - x(\mathbf{a_y}\cdot\mathbf{a_\varphi})$$

$$= -y\sin\varphi - x\cos\varphi = -\rho\sin^2\varphi - \rho\cos^2\varphi = -\rho$$

Thus,

$$B = -\rho\mathbf{a_\varphi} + z\mathbf{a_z} \quad \text{(Ans)}$$

Problem-1.10 (b):

Transform the vector field $G = \left(\frac{xz}{y}\right) \mathbf{a_x}$ into spherical components and variables.

Solution:

We find the three spherical components by dotting G with appropriate unit vectors and we change variables during the procedure:

$$G_r = G \cdot \mathbf{a_r} = \frac{xz}{y} \mathbf{a_x} \cdot \mathbf{a_r} = \frac{xz}{y} \sin\theta \cos\varphi$$

$$= r \sin\theta \cos\theta \frac{\cos^2\varphi}{\sin\varphi}$$

$$G_\theta = G \cdot \mathbf{a_\theta} = \frac{xz}{y} \mathbf{a_x} \cdot \mathbf{a_\theta} = \frac{xz}{y} \cos\theta \cos\varphi$$

$$= r \cos^2\theta \frac{\cos^2\varphi}{\sin\varphi}$$

$$G_\varphi = G \cdot \mathbf{a_\varphi} = \frac{xz}{y} \mathbf{a_x} \cdot \mathbf{a_\varphi} = \frac{xz}{y} (-\sin\varphi)$$

$$= -r \cos\theta \cos\varphi$$

Collecting these results, we have

$$G = r \cos\theta \cos\varphi \,(\sin\theta \cot\varphi \,\mathbf{a_r} + \cos\theta \cot\varphi \,\mathbf{a_\theta} - \mathbf{a_\varphi}) \quad \text{(Ans)}$$