INTRODUCTION TO DIFFERENTIAL EQUATIONS

Many physical problems, when formulated in mathematical forms, lead to differential equations. Differential equations enter naturally as models for many phenomena in economics, commerce, engineering etc. Many of these phenomena are complex in nature and very difficult to understand. But when they are described by differential equations, it is easy to analyses them. For example, if the rate of change of cost for x outputs is directly proportional to the cost, then this phenomenon is described by the differential equation,

$$\frac{dC}{dx} = k\ C,$$

Where C is the cost and k is constant. The solution of this differential equation is
$C = C_0\, e^{kx}$ where $C = C_0$ when $x = 0$.

EXAMPLE 1. *Mechanical Oscillator.* Consider a block of mass m lying on a table and restrained laterally by an ordinary coil spring (Fig. 1), and denote by x the displacement of the mass (measured as positive to the right) from its "equilibrium position;" that is, when $x = 0$ the spring is neither stretched nor compressed. We imagine the mass to be disturbed from its equilibrium position by an initial disturbance and/or an applied force $F(t)$, where t is the time, and we seek the differential equation governing the resulting displacement history $x(t)$.

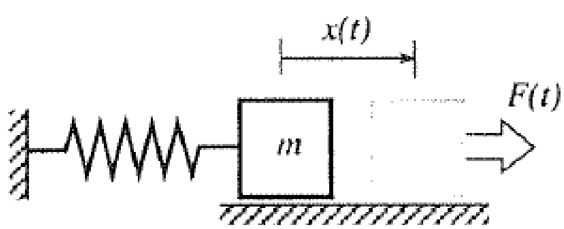
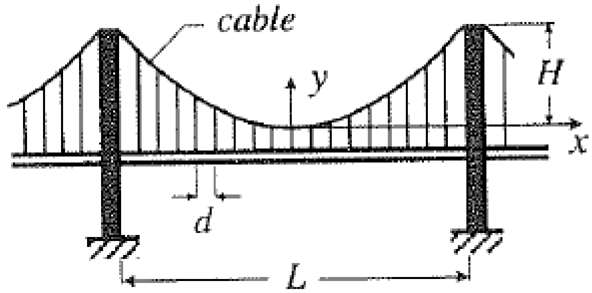


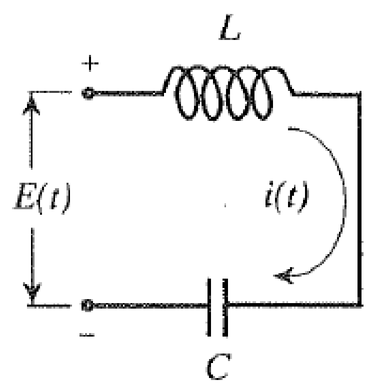
Figure 1. Mechanical oscillator.

EXAMPLE 2. *Suspension Bridge Cable.* To design a suspension bridge cable, one needs to know the relationships among the deflected shape of the cable, the tension in it, and the weight distribution that it needs to support.

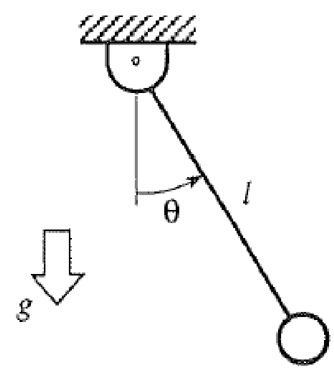
In the case of a typical suspension bridge, the roadbed supported by the cables is much heavier than the cables themselves, so let us neglect the weight of the cables, and assume that the loading is due entirely to the roadbed. Consider the configuration shown



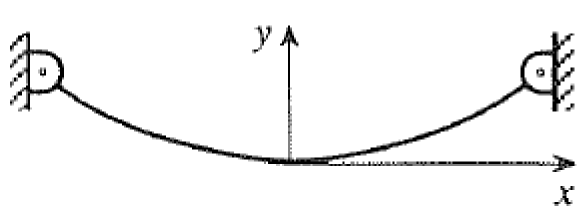
Example 3 $$L\frac{d^2i}{dt^2}+\frac{1}{C}i=\frac{dE}{dt},$$



Example 4 $$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0,$$



Example 5 $$\frac{d^2y}{dx^2}=C\sqrt{1+\left(\frac{dy}{dx}\right)^2},$$



FORMATION OF DIFFERENTIAL EQUATIONS

A Differential Equation is one which involves one or more independent variables, a dependent variable and one or more of their differential coefficients. There are two types of differential equations:

(i) Ordinary differential equations involving only one independent variable and derivatives of the dependent variable with respect to the independent variable.

(ii) Partial differential equations which involve more than one independent variable and partial derivatives of the dependent variable with respect to the independent variables.

The following are a few examples for differential equations:

1) $\left(\frac{dy}{dx}\right)^2 - 3\frac{dy}{dx} + 2y = e^x$ 2) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$

3) $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = k\frac{d^2y}{dx^2}$ 4) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

5) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ 6) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y$

(1), (2) and (3) are ordinary differential equations and (4), (5) and (6) are partial differential equations.

ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

The order of the derivative of the highest order present in a differential equation is called the order of the differential equation. For example, consider the differential equation

$$x^2\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{d^3y}{dx^3}\right)^2 + 7\frac{dy}{dx} - 4y = 0$$

The orders of $\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ are 3, 2 and 1 respectively.

So the highest order is 3. Thus the order of the differential equation is 3.

The degree of the derivative of the highest order present in a differential equation is called the degree of the differential equation. Here the differential coefficients should be free from the radicals and fractional exponents. Thus the degree of

$$x^2\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{d^3y}{dx^3}\right)^2 + 7\frac{dy}{dx} - 4y = 0 \quad \text{is } 2$$

Example 1

Write down the order and degree of the following differential equations.

(i) $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right) + y = 3e^x$ (ii) $\left(\frac{d^2y}{dx^2}\right)^3 + 7\left(\frac{dy}{dx}\right)^4 = 3\sin x$

(iii) $\frac{d^2x}{dy^2} + a^2x = 0$ (iv) $\left(\frac{dy}{dx}\right)^2 - 3\frac{d^3y}{dx^3} + 7\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right) - \log x = 0$

(v) $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 4x$ (vi) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{2}{3}} = \frac{d^2y}{dx^2}$

(vii) $\frac{d^2y}{dx^2} - \sqrt{\frac{dy}{dx}} = 0$ (viii) $\sqrt{1+x^2} = \frac{dy}{dx}$

Solution :

The order and the degree respectively are,

(i) 1 ; 3 (ii) 2 ; 3 (iii) 2 ; 1 (iv) 3 ; 1

(v) 1 ; 2 (vi) 2 ; 3 (vii) 2 ; 2 (viii) 1 ; 1

FAMILY OF CURVES

Sometimes a family of curves can be represented by a single equation. In such a case the equation contains an arbitrary constant c. By assigning different values for c, we get a family of curves. In this case c is called the parameter or arbitrary constant of the family.

Examples

(i) $y = mx$ represents the equation of a family of straight lines through the origin , where m is the parameter.

(ii) $x^2 + y^2 = a^2$ represents the equation of family of concentric circles having the origin as centre, where a is the parameter.

(iii) $y = mx + c$ represents the equation of a family of straight lines in a plane, where m and c are parameters.

FORMATION OF ORDINARY DIFFERENTIAL EQUATION

Example 2

Form the differential equation of the family of curves y = A cos 5x + B sin 5x where A and B are parameters.

Solution :

Given $y = \text{A} \cos 5x + \text{B} \sin 5x$

$$\frac{dy}{dx} = -5\text{A} \sin 5x + 5\text{B} \cos 5x$$

$$\frac{d^2y}{dx^2} = -25\ (A \cos 5x) - 25\ (B \sin 5x) = -25y$$

$$\therefore \quad \frac{d^2y}{dx^2} + 25y = 0.$$

Example 3

Form the differential equation of the family of curves $y = ae^{3x} + be^{x}$ where a and b are parameters.

Solution :

$$y = ae^{3x} + be^{x} \quad \text{------------(1)}$$

$$\frac{dy}{dx} = 3ae^{3x} + be^{x} \quad \text{------------(2)}$$

$$\frac{d^2y}{dx^2} = 9ae^{3x} + be^{x} \quad \text{------------(3)}$$

$$(2) - (1) \Rightarrow \frac{dy}{dx} - y = 2ae^{3x} \quad \text{------------(4)}$$

$$(3) - (2) \Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 6ae^{3x} = 3\left(\frac{dy}{dx} - y\right) \quad \text{[using (4)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

Example 4

Find the differential equation of a family of curves given by $y = a \cos (mx + b)$, a and b being arbitrary constants.

Solution :

$$y = a \cos (mx + b) \quad \text{------------(1)}$$

$$\frac{dy}{dx} = -ma \sin (mx + b)$$

$$\frac{d^2y}{dx^2} = -m^2 a \cos (mx + b) = -m^2 y \quad \text{[using (1)]}$$

$$\therefore \quad \frac{d^2y}{dx^2} + m^2 y = 0 \text{ is the required differential equation.}$$

Example 5

Find the differential equation by eliminating the arbitrary constants a and b from y = a tan x + b sec x.

Solution :

y = a tan x + b sec x

Multiplying both sides by cos x we get,

y cos x = a sin x + b

Differentiating with respect to x we get

$$y(-\sin x) + \frac{dy}{dx}\cos x = a\cos x$$

$$\Rightarrow \quad -y\tan x + \frac{dy}{dx} = a \qquad \text{-----------(1)}$$

Differentiating (1) with respect to x, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx}\tan x - y\sec^2 x = 0$$

PROBLEM LIST

1) Find the order and degree of the following :

(i) $x^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = \cos x$ (ii) $\frac{d^3y}{dx^3} - 3\left(\frac{d^2y}{dx^2}\right)^2 + 5\frac{dy}{dx} = 0$

(iii) $\frac{d^2y}{dx^2} - \sqrt{\frac{dy}{dx}} = 0$ (iv) $\left(1 + \frac{d^2y}{dx^2}\right)^{\frac{1}{2}} = \frac{dy}{dx}$

(v) $\left(1 + \frac{dy}{dx}\right)^{\frac{1}{3}} = \frac{d^2y}{dx^2}$ (vi) $\sqrt{1 + \frac{d^2y}{dx^2}} = x\frac{dy}{dx}$

(vii) $\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}} = \left(\frac{dy}{dx}\right)^2$ (viii) $3\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 3y = e^x$

(ix) $\frac{d^2y}{dx^2} = 0$ (x) $\left(\frac{d^2y}{dx^2} + 1\right)^{\frac{2}{3}} = \left(\frac{dy}{dx}\right)^{\frac{1}{3}}$

2) Find the differential equation of the following

(i) $y = mx$ (ii) $y = cx - c + c^2$

(iii) $y = mx + \frac{a}{m}$, where m is arbitrary constant

(iv) $y = mx + c$ where m and c are arbitrary constants.

3) Form the differential equation of family of rectangular hyperbolas whose asymptotes are the coordinate axes.

4) Find the differential equation of all circles $x^2 + y^2 + 2gx = 0$ which pass through the origin and whose centres are on the x-axis.

5) Form the differential equation of $y^2 = 4a(x + a)$, where a is the parameter.

6) Find the differential equation of the family of curves $y = ae^{2x} + be^{3x}$ where a and b are parameters.

7) Form the differential equation for $y = a\cos 3x + b\sin 3x$ where a and b are parameters.

8) Form the diffrential equation of $y = ae^{bx}$ where a and b are the arbitrary constants.

9) Find the differential equation for the family of concentric circles $x^2 + y^2 = a^2$, a is the paramter.