# Microwave Engineering

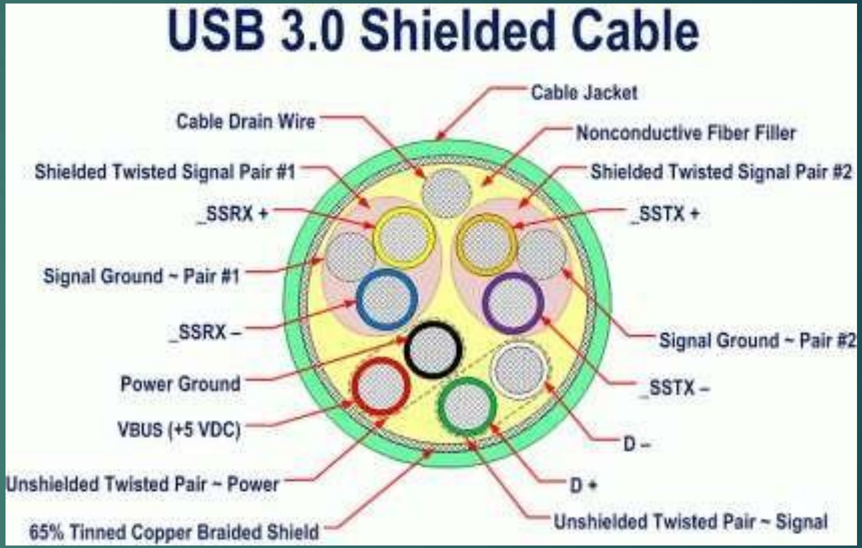
## ETE 415

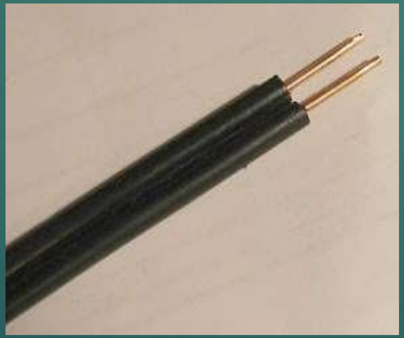
### LECTURE 2
### TRANSMISSION-LINE THEORY-I

# T-Line Theory

T-Lines are used in countless ways in modern socity

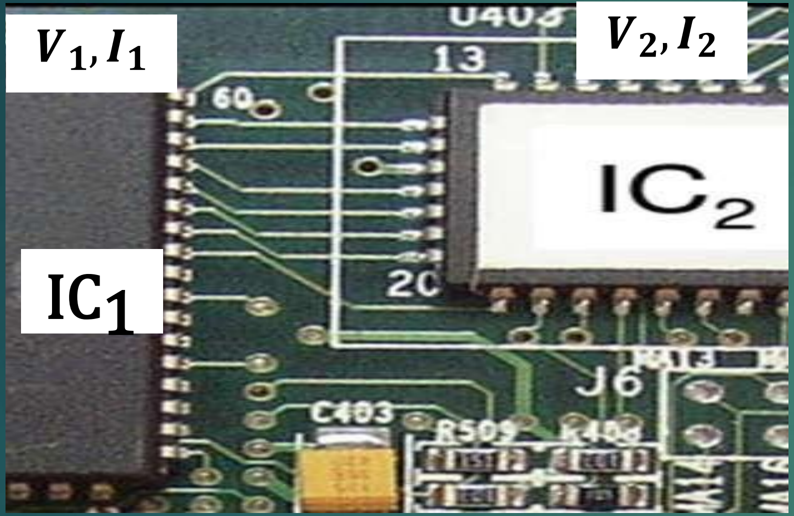
- Communications
- Electronics circuits/ Computers
- Power distribution
- Photonics

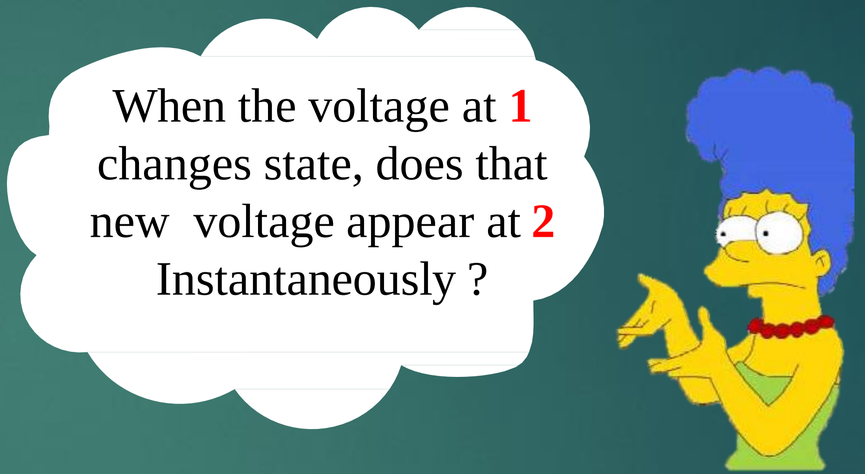



# Why T-Lines?-I

Consider two ICs being connected by conducting traces on PCB.




If these two points are separated by a large electrical distance, there will be a **propagation delay** as the change in state (electrical signal) travels to 2. $\Rightarrow$ Not an instantaneous effect!!

This propagation of voltage signals is modeled as a **"T-line"**.

We will see that voltage and current can propagate along a T-Line as **waves**! Fantastic.

# T-Line Theory

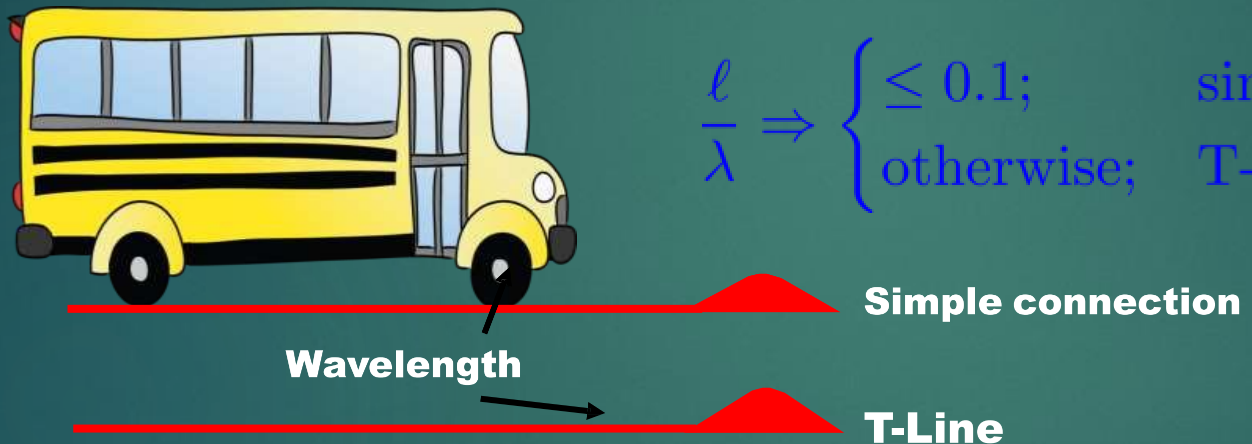
- Lumped circuits: resistors, capacitors, inductors

**Neglects time delays (phase change)**

- Distributed circuit elements: T-lines

**Account for propagation and time delays (phase change)**

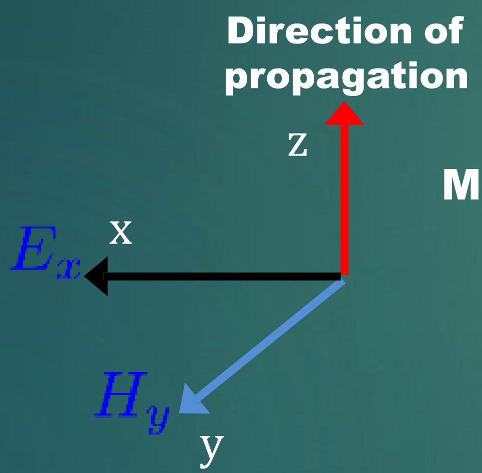
$$\frac{\ell}{\lambda} \Rightarrow \begin{cases} \leq 0.1; & \text{simple connection} \\ \text{otherwise;} & \text{T-Line,} \end{cases}$$

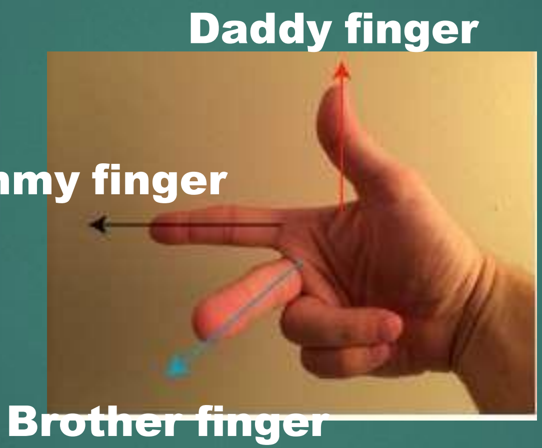


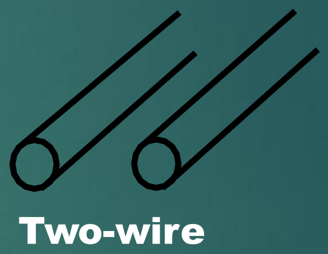

We need T-line theory whenever the length of a line is significant compared to a wavelength.

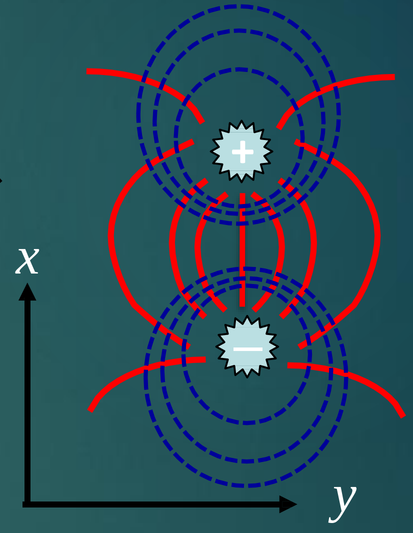
# TEM T-Lines

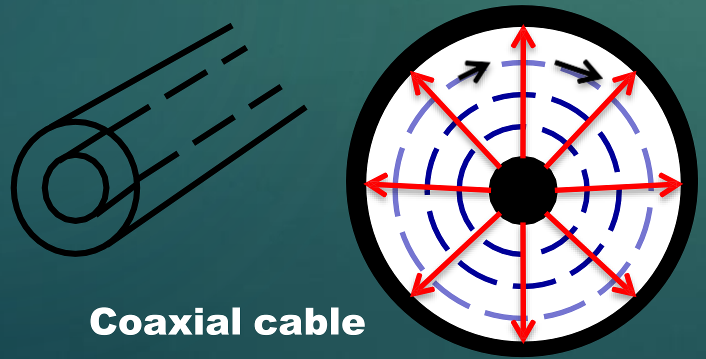
- All true TLs share one common characteristic: the *E* and *H* fields are all ⊥ the direction of propagation.
- These are called **TEM fields** for **t**ransverse **e**lectric and **m**agnetic fields.

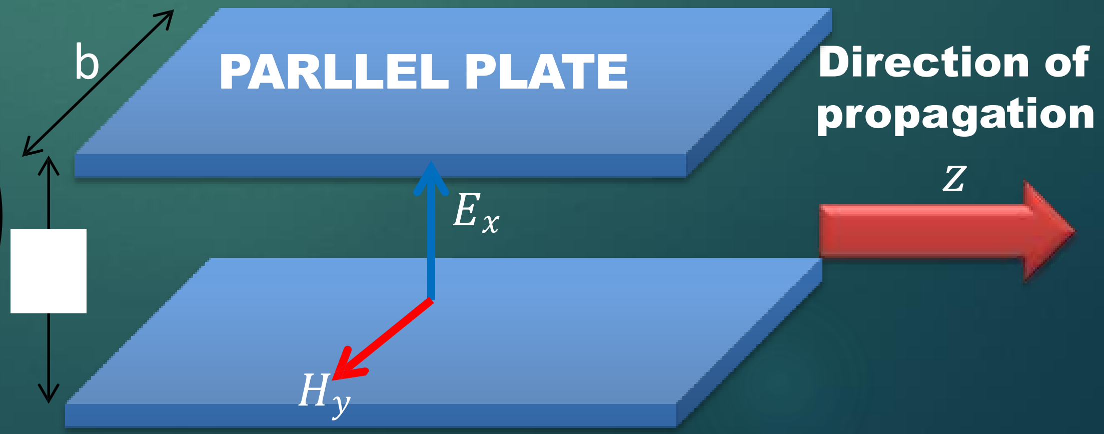


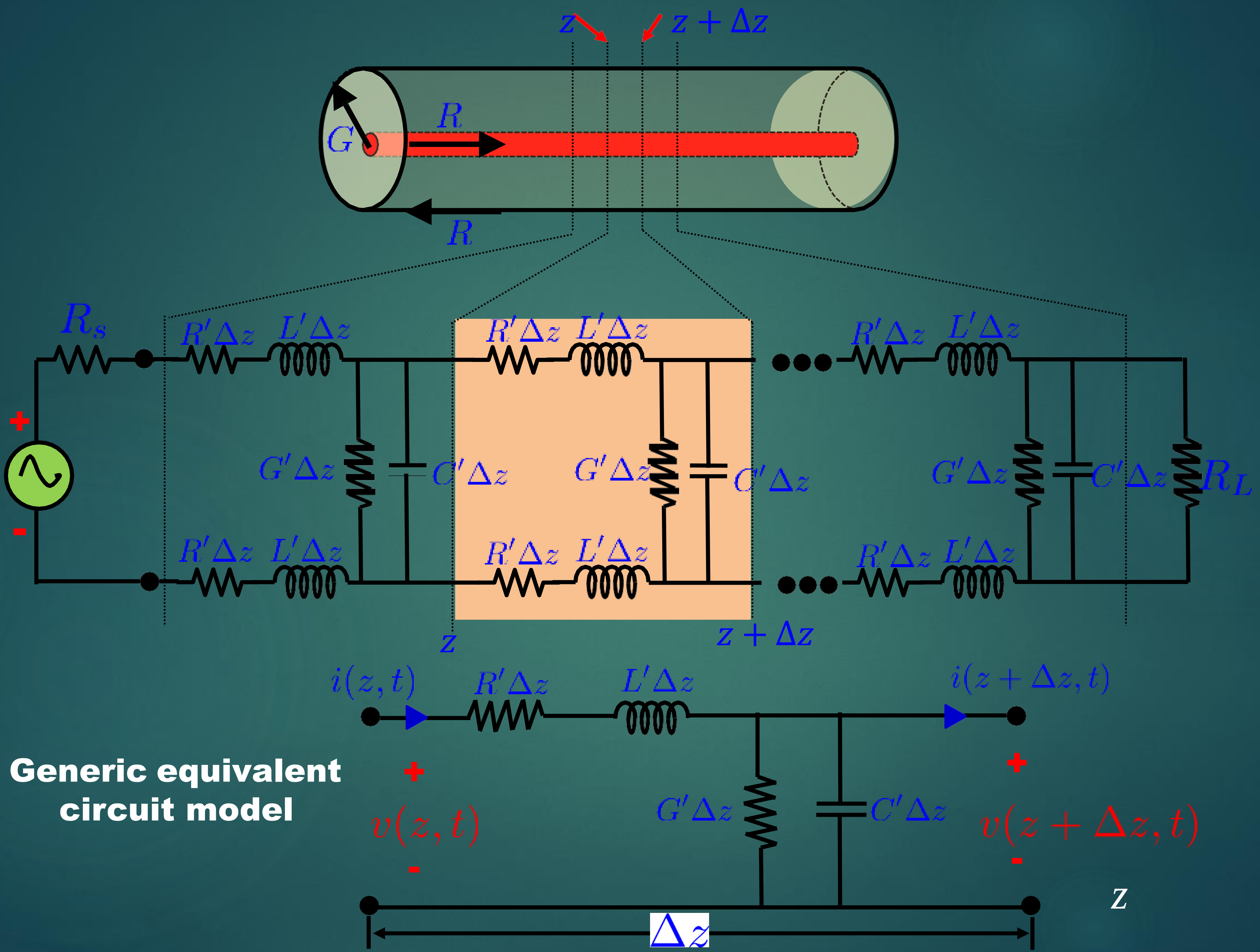



# T-Line Model ($RLGC$ Model)

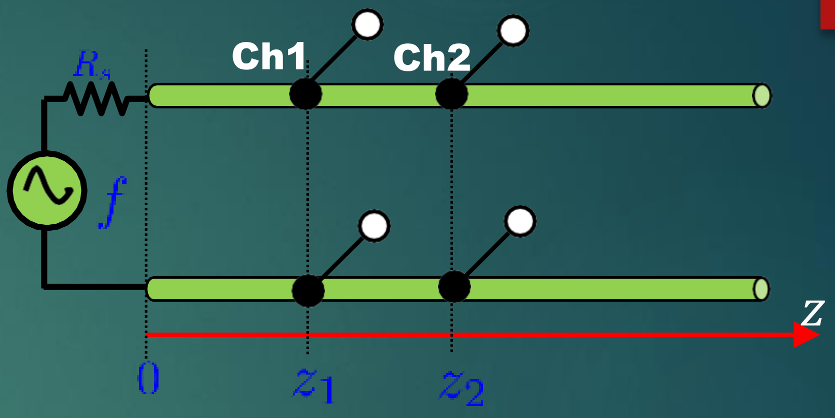
- There is **conduction current** in the two conductors and a **displacement current** between these two conductors where the electric field $E$ is varying with time.
- Conduction current impedance effects:
  - $\boldsymbol{R'}$ [Ω/m], series resistance due to losses in the conductors,
  - $\boldsymbol{L'}$ [H/m], series inductance due to the current flow in the conductors and the magnetic flux linking the current path.
- Displacement current impedance effects:
  - $\boldsymbol{G'}$ [S/m], shunt conductance due to losses in the dielectric between the conductors,
  - $\boldsymbol{C'}$ [F/m], shunt capacitance due to the time varying electric field between the two conductors.

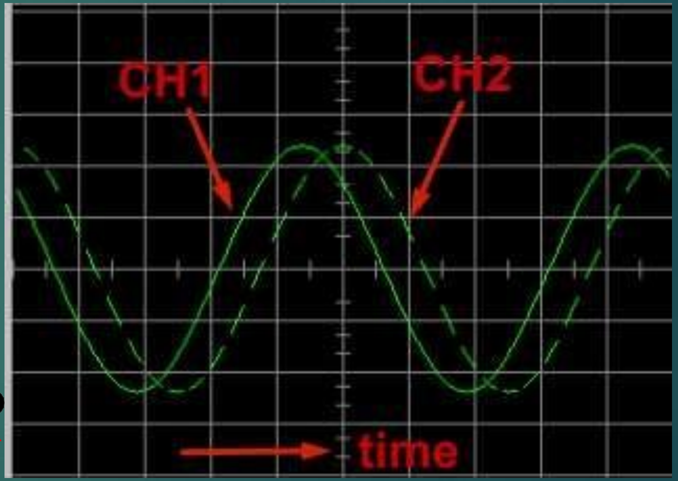
Note: $R'$ and $G'$, represent loss.

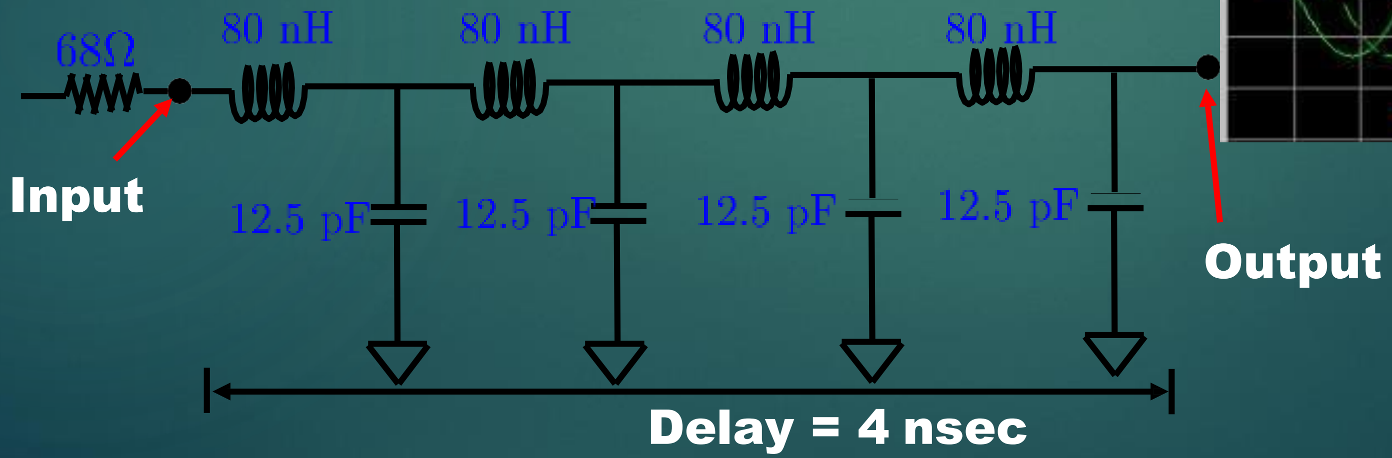



**Generic equivalent circuit model**

# A 4 nsec, 80 Ω, lumped element T-Line

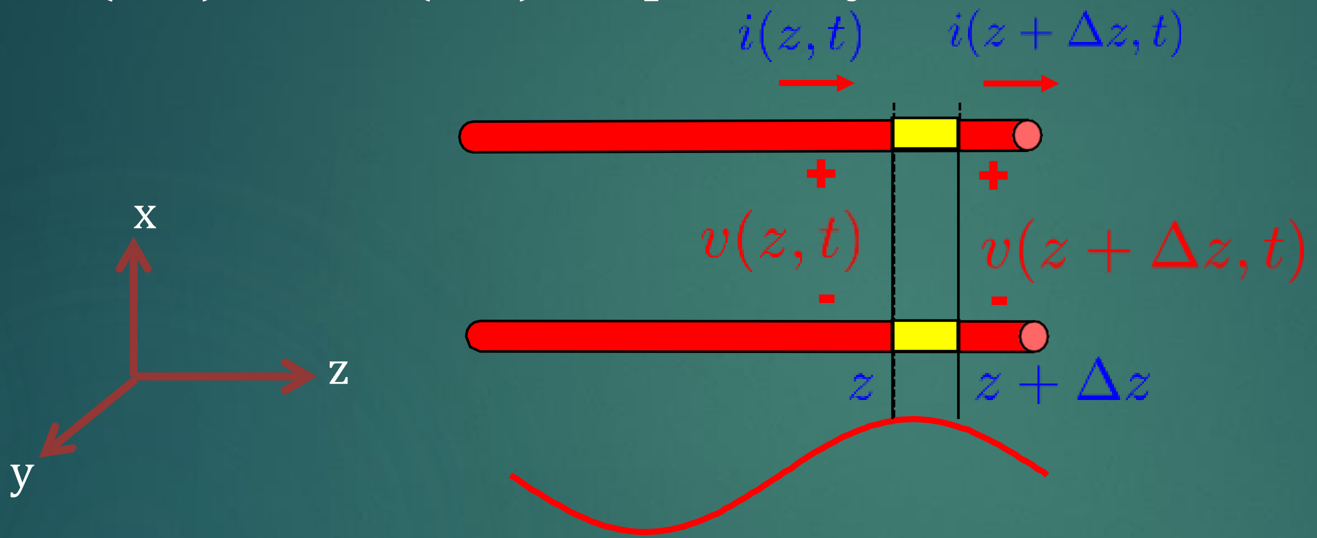


# Analysis of T-Lines

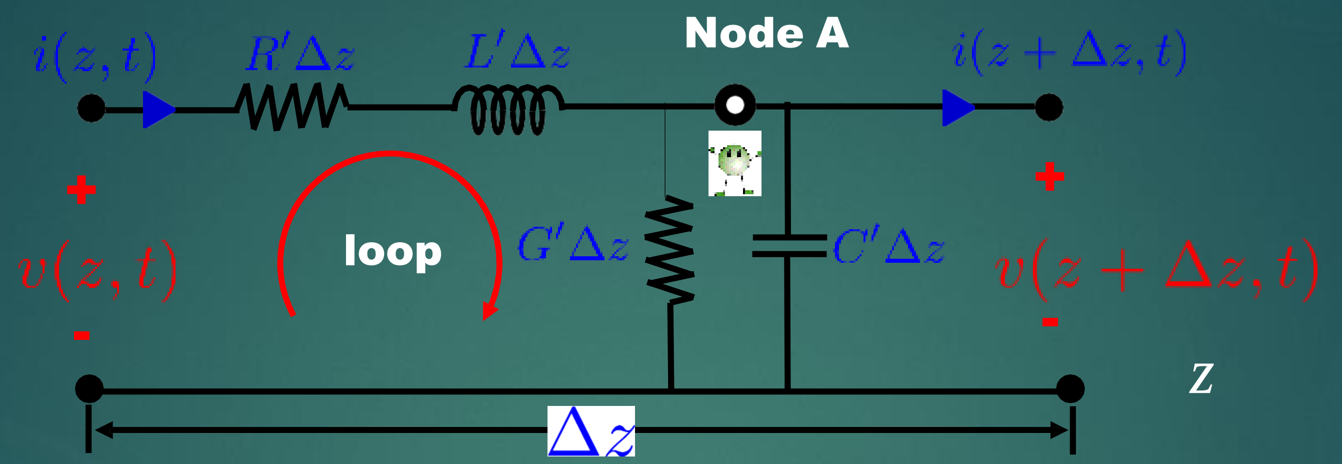
On a T-Line, the **voltage and current vary along the structure in time $t$ and spatially in the $z$ direction** ⇒ they are expressed as $i(z, t)$, and $v(z, t)$, respectively




How do we solve for $v(z, t)$ and $i(z,t)$?

- We first need to develop the governing equations for the voltage and current, and then solve these equations.

# T-Line Equations-I




- To develop the governing equation for $v(z, t)$, apply **KVL**:

$$v(z,t) = v(z+\Delta z,t) + i(z,t)R'\Delta z + L'\Delta z\frac{\partial i(z,t)}{\partial t} \quad (1a)$$

$$\Delta v = v(z+\Delta z,t) - v(z,t) = -i(z,t)R'\Delta z - L'\Delta z\frac{\partial i(z,t)}{\partial t}$$

- Similarly, for the current $i(z, t)$ apply **KCL** at the node A:

$$i(z,t) = i(z+\Delta z,t) + v(z+\Delta z,t)G'\Delta z + C'\Delta z\frac{\partial v(z+\Delta z,t)}{\partial t} \quad (1b)$$

$$\Delta i = i(z+\Delta z,t) - i(z,t) = -v(z+\Delta z,t)G'\Delta z - C'\Delta z\frac{\partial v(z+\Delta z,t)}{\partial t}$$

# T-Line Equations-II

- Divide (1a) and (1b) by $\Delta z$:

$$\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = -R'i(z,t) - L'\frac{\partial i(z,t)}{\partial t} \qquad (2)$$

$$\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = -G'v(z+\Delta z,t) - C'\frac{\partial v(z+\Delta z,t)}{\partial t} \qquad (3)$$

- In the limit as $\Delta z \rightarrow 0$, the term on the LHS in (2) and (3) is the **forward difference definition of derivative**. Hence,

$$\frac{\partial v(z,t)}{\partial z} = -R'i(z,t) - L'\frac{\partial i(z,t)}{\partial t} \qquad (4)$$

$$\frac{\partial i(z,t)}{\partial z} = -G'v(z,t) - C'\frac{\partial v(z,t)}{\partial t} \qquad (5)$$

**coupled partial differential equations**

- Eqs. (4) and (5) are called the **telegrapher equations** or the **T-Line equations**.

# T-Line Wave Equations-I

- Now combine (4) and (5) to form two equations, each a **function of $v$ or $i$ only**.
- To do this, take $\frac{\partial}{\partial}$ of Eq. (4) $\left(\frac{\partial v}{\partial z} = -R' i - L' \frac{\partial i}{\partial t}\right)$:

**Switch the order of the derivatives.**

$$\frac{\partial^2 v}{\partial z^2} = -R' \frac{\partial i}{\partial z} - L' \frac{\partial}{\partial z}\left(\frac{\partial i}{\partial t}\right) = -R' \frac{\partial i}{\partial z} - L' \frac{\partial}{\partial t}\left(\frac{\partial i}{\partial z}\right)$$

- Substituting from Eq. (5) $\left(\frac{\partial i}{\partial z} = -G' v - C' \frac{\partial v}{\partial t}\right)$:

$$\frac{\partial^2 v}{\partial z^2} = -R'\left[-G' v - C' \frac{\partial v}{\partial t}\right] - L'\left[-G' \frac{\partial v}{\partial t} - C' \frac{\partial^2 v}{\partial t^2}\right]$$

$$\frac{\partial^2 v}{\partial z^2} - (R'G')v - (R'C' + L'G')\frac{\partial v}{\partial t} - L'C' \frac{\partial^2 v}{\partial t^2} = 0$$

- The same equation also holds for $i$.

$$\frac{\partial^2 i}{\partial z^2} - (R'G')i - (R'C' + L'G')\frac{\partial i}{\partial t} - L'C' \frac{\partial^2 i}{\partial t^2} = 0$$

# T-Line Wave Equations-II

- Time-Harmonic Waves: $\frac{\partial}{\partial t} \rightarrow jw$

$$\frac{d^2V}{dz^2} - (R'G')V - (R'C' + L'G')\,jw\,V - L'C'(-w^2)V = 0$$

**Convention:** small letter for time-domain form, capital letter for phasor.

- Note that

$$R'G' + jw(R'C' + L'G') - w^2L'C' = (R' + jwL')(G' + jwC')$$

- Then we can write

$$\frac{d^2V}{dz^2} - \gamma^2 V = 0 \qquad \text{and} \qquad \frac{d^2I}{dz^2} - \gamma^2 I = 0$$

  - Where

$$\gamma = \alpha + j\beta = \sqrt{(R' + jwL')(G' + jwC')}$$

  - is the complex propagation constant, which is a function of frequency.
  - $\alpha \geq 0$ is the attenuation constant and $\beta \geq 0$ is the phase constant.

# Voltage Wave Equation Solutions

- The voltage wave equation has solution in the form

$$V(z) = \underbrace{V_o^+ e^{-\gamma z}}_{V^+(z)} + \underbrace{V_o^- e^{\gamma z}}_{V^-(z)}$$

+z **wave** −z **wave**

Note: $V_o^{\mp} = |V_o^{\mp}| e^{j\phi^{\mp}}$

**Complex**

  - where $\phi^{\mp}$ is the phase angle of the complex voltage $V_o^{\mp}$

- **Forward** travelling wave (a wave traveling in the $+z$ direction):

$$V^+(z) = V_o^+ e^{-\gamma z} = V_o^+ e^{-\alpha z} e^{-j\beta z}$$

$$\gamma = \alpha + j\beta$$

- Converting back to the time domain

$$v^+(z,t) = \Re\left\{\left(V_o^+ e^{-\alpha z} e^{-j\beta z}\right) e^{jwt}\right\}$$

$$= \Re\left\{\left(|V_o^+| e^{j\phi^+} e^{-\alpha z} e^{-j\beta z}\right) e^{jwt}\right\}$$

$$= |V_o^+| e^{-\alpha z} \cos(wt - \beta z + \phi^+)$$

# Voltage Wave Equation Solutions

- **Backward** travelling wave (a wave traveling in the $-z$ direction):

$$V^-(z) = V_o^- e^{+\gamma z} = V_o^- e^{+\alpha z} e^{+j\beta z}$$

- Converting back to the time domain

$$\begin{aligned} v^-(z,t) &= \Re\left\{\left(V_o^- e^{\alpha z} e^{j\beta z}\right) e^{jwt}\right\} \\ &= \Re\left\{\left(|V_o^-| e^{j\phi^-} e^{\alpha z} e^{j\beta z}\right) e^{jwt}\right\} \\ &= |V_o^-| e^{+\alpha z} \cos(wt + \beta z + \phi^-) \end{aligned}$$

  - where $\phi^-$ is the phase angle of the complex voltage $V_o^-$

- Complete time domain solution:

$$\begin{aligned} v(z,t) &= \Re\{V(z) e^{jwt}\} \\ &= |V_o^+| e^{-\alpha z} \cos(wt - \beta z + \phi^+) + |V_o^-| e^{+\alpha z} \cos(wt + \beta z + \phi^-) \end{aligned}$$

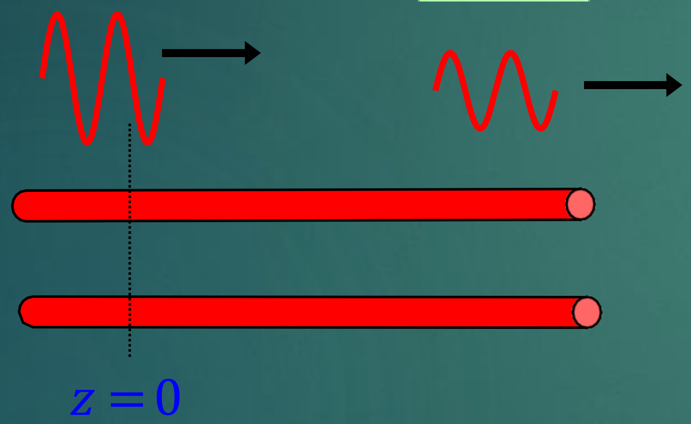
$$i(z,t) = |I_o^+| e^{-\alpha z} \cos(wt - \beta z + \phi^+) + |I_o^-| e^{+\alpha z} \cos(wt + \beta z + \phi^-)$$

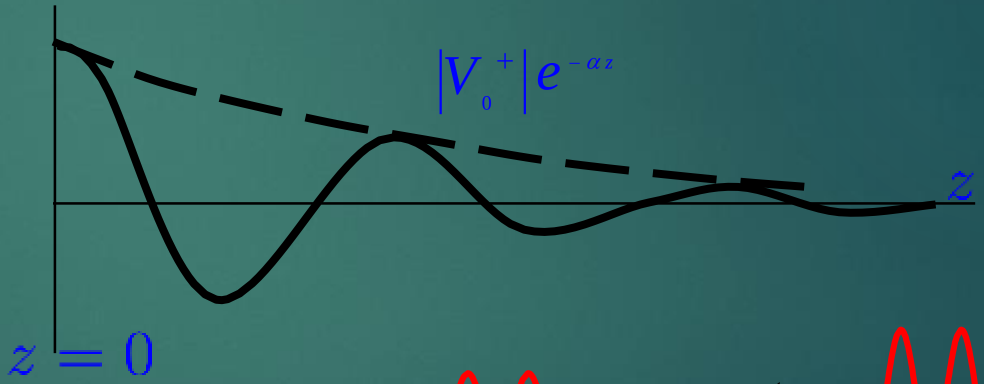
# Attenuation Constant ($\alpha$)

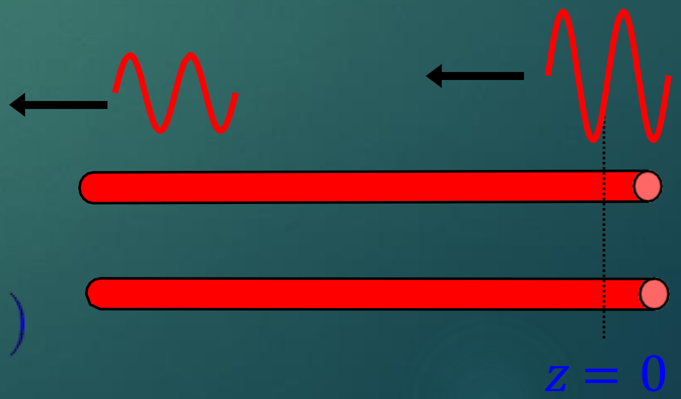
- **The attenuation constant $\alpha$ [Np/m]:** decreases the amplitude of the voltage and current wave along the T-Line.
- For +ve traveling wave:

$$1\ \text{Np} = 10\log e^2 = 8.686\ \text{dB}$$

$$v^+(z,t) = |V_o^+|\, e^{-\alpha z} \cos(wt - \beta z + \phi^+)$$



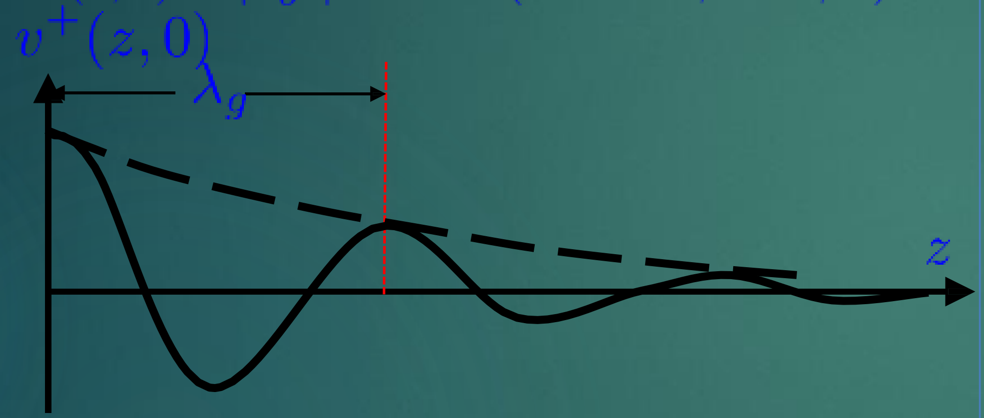

- For –ve traveling wave:

$$v^-(z,t) = |V_o^-|\, e^{+\alpha z} \cos(wt + \beta z + \phi^-)$$

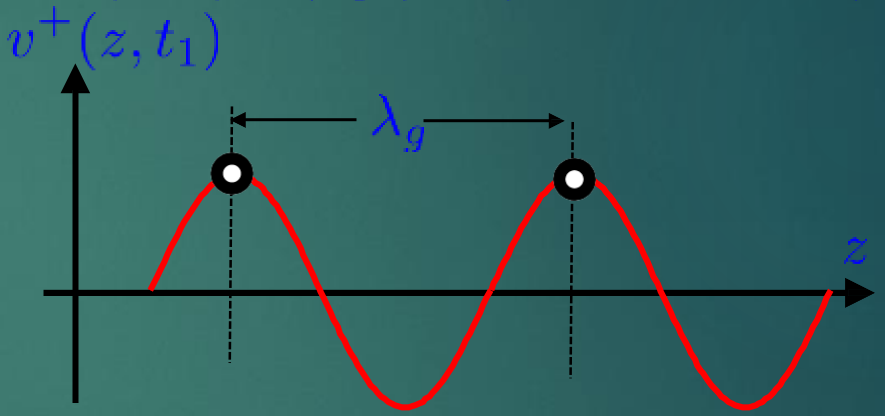
# Wavelength ($\lambda_g$)

- **The wavelength, $\lambda_g$:**- the distance between two successive maxima (or minima, or any other reference points) on the wave at a fixed instant of time.

$$v^+(z,0) = |V_o^+|e^{-\alpha z}\cos(w\times 0 - \beta z + \phi^+)$$




$$v^+(z,t_1) = |V_o^+|\cos(wt_1 - \beta z + \phi^+)$$




- Thus,

$$(wt_1 - \beta z + \phi^+) - [wt_1 - \beta(z + \lambda_g) + \phi^+] = 2\pi$$

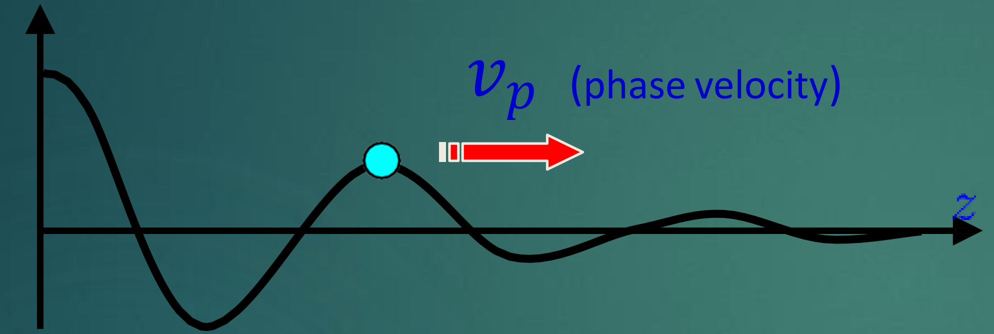
- The wave "repeats" when: $\beta\lambda_g = 2\pi$
- Hence:

$$\lambda_g = \frac{2\pi}{\beta}$$

# Phase Velocity ($v_p$)

- Phase velocity $v_p$: velocity at which a fixed phase point on the wave travels.

$$v^+(z,0) = |V_o^+|e^{-\alpha z}\cos(w \times 0 - \beta z + \phi^+)$$




- Consider **"riding"** one part of the wave

$$wt - \beta z = \text{constant (choose 0)} \quad \Rightarrow \quad z = \frac{wt}{\beta}$$

- **Phase velocity** calculation

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{wt}{\beta}\right) = \frac{w}{\beta}$$

- Hence $\boxed{v_p = \frac{w}{\beta}}$

**In expanded form:**

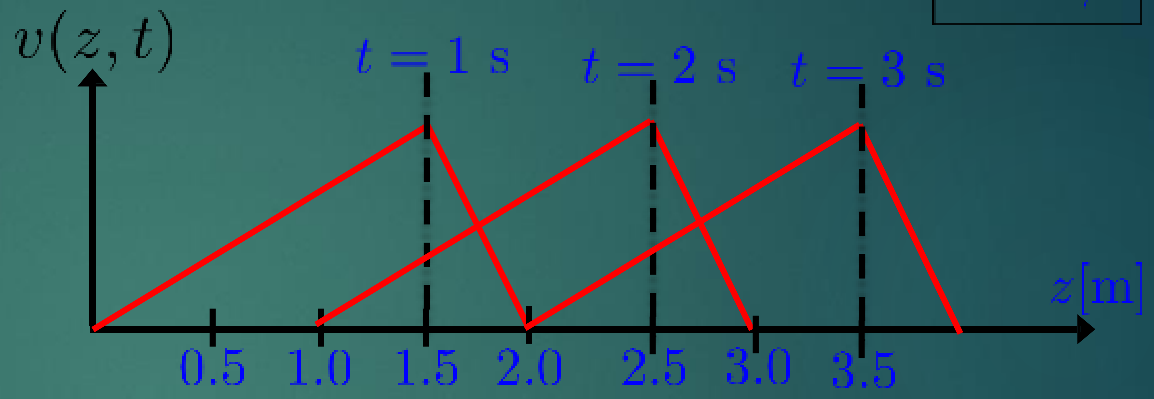
$$v_p = \frac{w}{\mathrm{Im}\{\sqrt{(R' + jwL')(G' + jwC')}\}}$$

# Wave Motion

$$v^+(z,t) = |V_o^+|\cos(wt - \beta z + \phi^+) = |V_o^+|\cos\left(w\left(t - \frac{\beta}{w}z\right) + \phi^+\right)$$

$$\boxed{v_p = \frac{w}{\beta}}$$

$$v^+(z,t) = f\left(t - \frac{z}{v_p}\right)$$




Let $v_p = 1$ m/s

**At $t = 1$ s**, focus on the peak located at $z = 1.5$ m

$$\Longrightarrow \; S_+ = t - \frac{z}{v_p} = 1 - \frac{1.5}{1} = -0.5$$

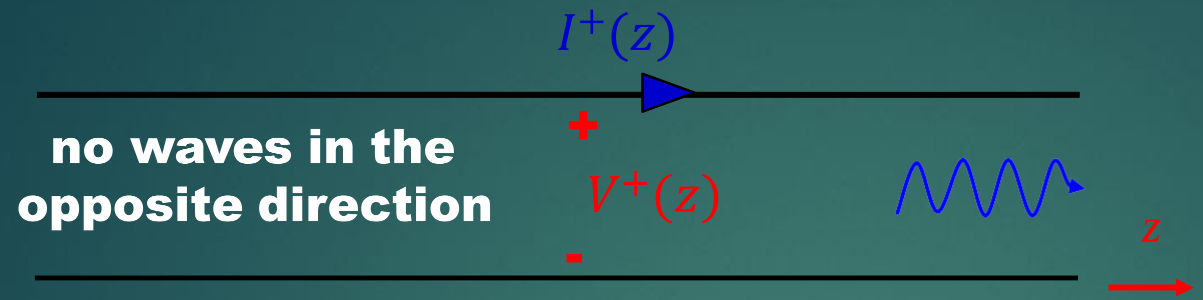
The argument $S_+$ stays constant for varying $t$ & $z \Longrightarrow$ **at $t = 2$ s**, for example:

$$S_+ = -0.5 = t - \frac{z}{v_p} = 2 - \frac{z}{1} \quad \Longrightarrow \quad z = 2.5 \text{ m.}$$

So, the peak has now moved to position $z = 2.5$ m at $t = 2$ s.
Likewise, every point on this function moves the same distance (1 m) in this time (1 s). $\Longrightarrow$ This is called **wave motion**.

The speed of this movement is: $\frac{\Delta z}{\text{m}} = \frac{1}{} = 1$ m s $= v_p$

# Characteristic Impedance ($Z_0$)




$$V^+(z) = V_o^+ e^{-\gamma z}$$
$$I^+(z) = I_o^+ e^{-\gamma z}$$

A wave is traveling in the $+z$ direction.

$$Z_0 \equiv \frac{V^+(z)}{I^+(z)} = \frac{V_0^+ e^{-\gamma z}}{I_0^+ e^{-\gamma z}} = \frac{V_0^+}{I_0^+}$$

(Note: $Z_0$ is a number, not a function of $z$)

- Use Telegrapher's Equation:

$$\frac{\partial v}{\partial z} = -R'i - L'\frac{\partial i}{\partial t} \quad \Rightarrow \quad \frac{dV}{dz} = -R'I - jwL'I$$

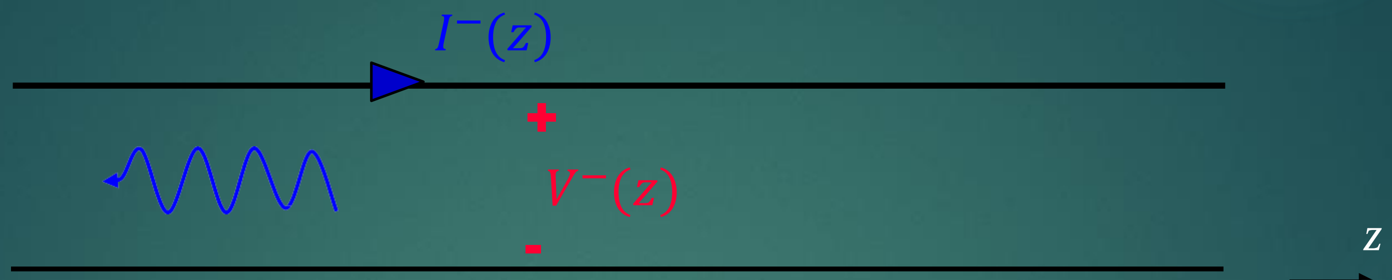
- Hence:

$$-\gamma V_0^+ e^{-\gamma z} = -(R' + jwL')I_0^+ e^{-\gamma z}$$

- From this we have:

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{R' + jwL'}{\gamma} = \sqrt{\frac{R' + jwL'}{G' + jwC'}}$$

# $Z_0$ for Backward-Traveling Wave

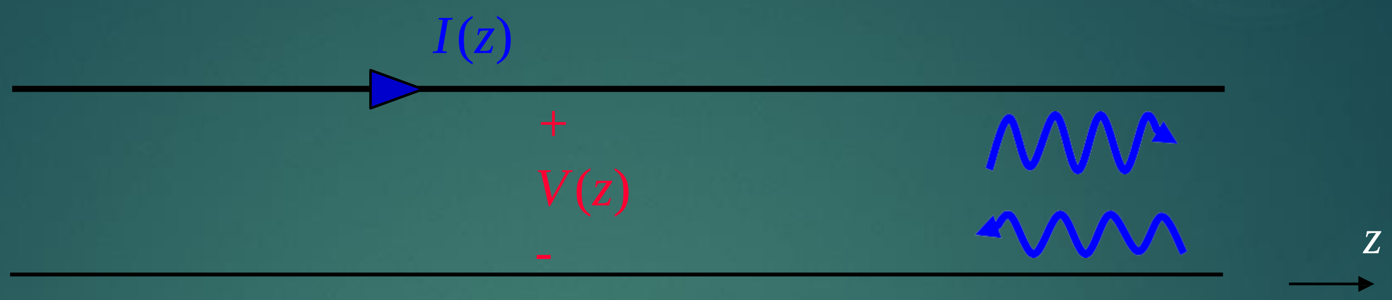



A wave is traveling in the **$-z$ direction**.

$$Z_0 = \frac{V^{-}(z)}{-I^{-}(z)} \quad \text{so} \quad \frac{V^{-}(z)}{I^{-}(z)} = -Z_0$$

Note: The reference directions for voltage and current are the same as for the forward wave.

# General Case

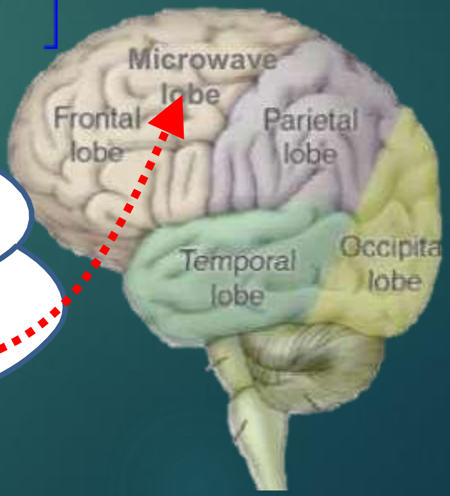



**Most general case:** A general superposition of forward and backward traveling waves:

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left[V_o^+ e^{-\gamma z} - V_o^- e^{+\gamma z}\right]$$

Both of these equations should be committed to memory. They are the general form of **phasor voltages** and **currents** on TLs



# 50 Ohm T-Lines!!

**In most RF systems,**

$Z_0 = 50\ \Omega$

**WHY???**

# The Lossless T-Line

- Set $R' = G' = 0$

$$\gamma = \alpha + j\beta = \sqrt{(R' + jwL')(G' + jwC')}$$

$$= jw\sqrt{L'C'} = j\beta$$

- So $\alpha = 0$ and $\beta = w\sqrt{L'C'}$

$$Z_0 = \sqrt{\frac{R' + jwL'}{G' + jwC'}} \Rightarrow Z_0 = \sqrt{\frac{L'}{C'}}$$

**real and independent of frequency**

$$v_p = \frac{w}{\beta} \Rightarrow v_p = \frac{1}{\sqrt{L'C'}}$$

**Independent of *f***

- If the medium between the two conductors is homogeneous (uniform) and is characterized by ($\mu$, $\varepsilon$), then we have that:

$$L'C' = \mu\varepsilon \quad \Rightarrow \quad v_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

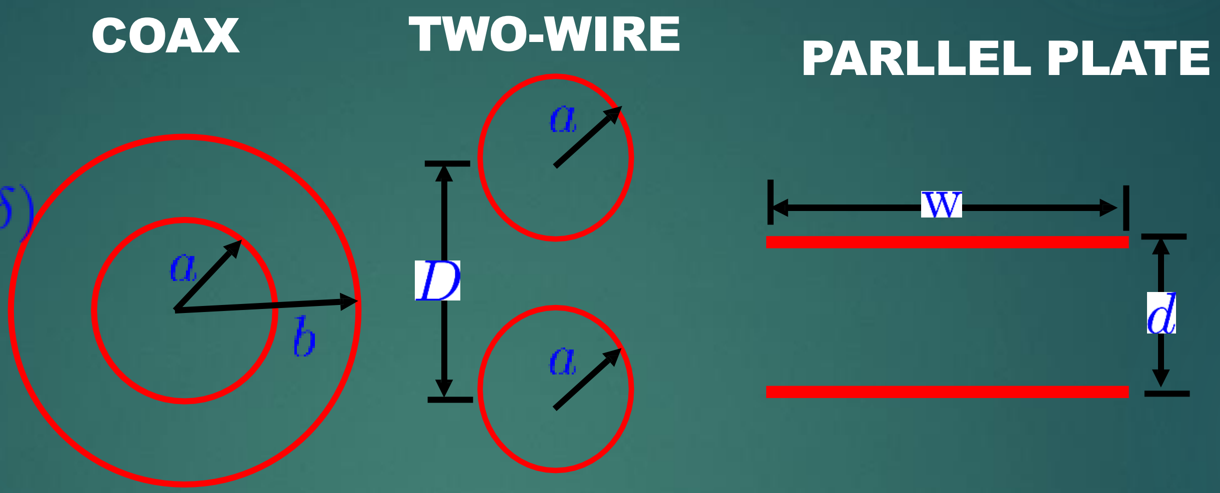
**it is always the speed of light (in the material).**

# Generality of T-Line Theory

$$R_s = \sqrt{\pi f \mu_c / \sigma_c}$$

$$\epsilon = \epsilon' - j\epsilon''$$
$$= \epsilon_o \epsilon_r (1 - j \tan \delta)$$

$$\delta = \sqrt{\frac{2}{w\mu\sigma}}$$

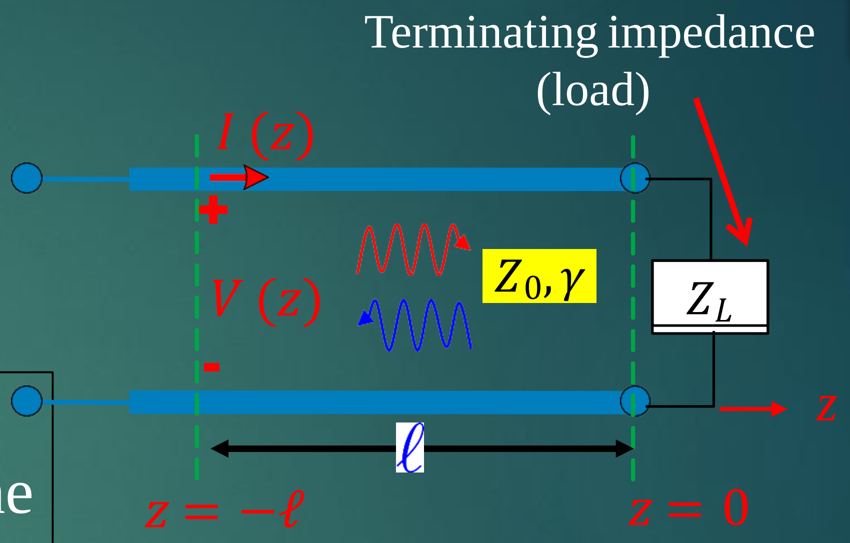



| | | | | |
|---|---|---|---|---|
| $L'$ | $\frac{\mu}{2\pi}\ln\frac{b}{a}$ | $\frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$ | $\frac{\mu d}{w}$ | H/m |
| $C'$ | $\frac{2\pi\epsilon'}{\ln(b\;\;a\;)}$ | $\frac{\pi\epsilon'}{\cosh^{-1}(D\;\;2a\;)}$ | $\frac{\epsilon' w}{d}$ | F/m |
| $R'$ | $\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$ | $\frac{R_s}{\pi a}$ | $\frac{2R_s}{w}$ | Ω/m |
| $G'$ | $\frac{2\pi\omega\epsilon''}{\ln(b\;\;a\;)}$ | $\frac{\pi\omega\epsilon''}{\cosh^{-1}(D\;\;2a\;)}$ | $\frac{\omega\epsilon'' w}{d}$ | S/m |

# Terminated T-Line-I

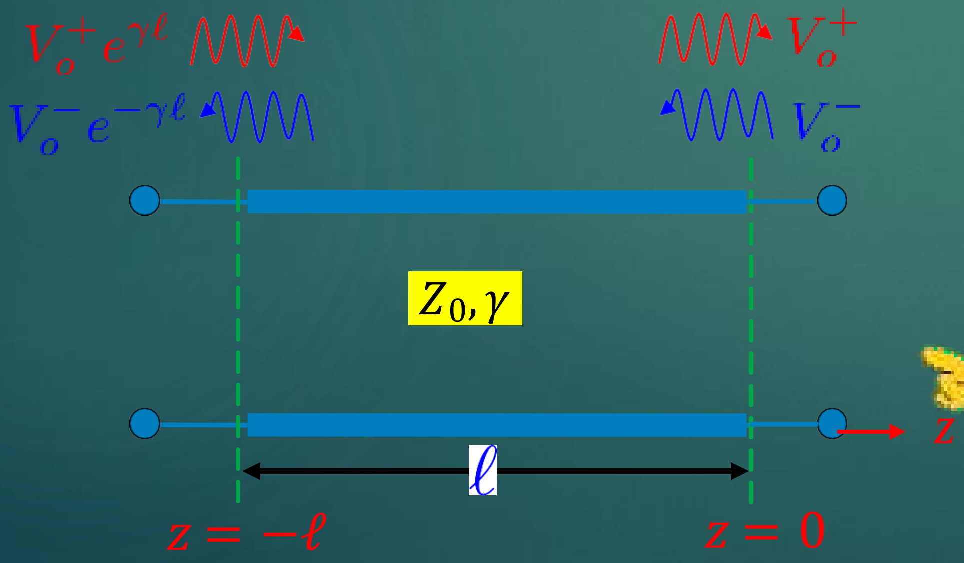
$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left[ V_o^+ e^{-\gamma z} - V_o^- e^{+\gamma z} \right]$$




**Where do we assign $z = 0$?**

For mathematical convenience, the usual choice is at the load, i.e., $z = 0$.




**Note:** The length $\ell$ measures distance from the load: $\ell = -z$

# Terminated T-Line-II

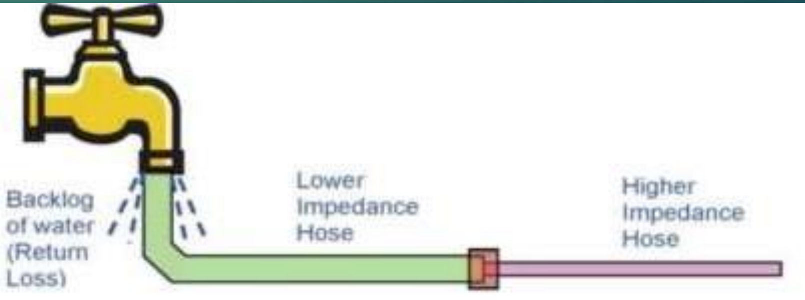
- The "lumped load" $Z_L$ that terminates the TL is considered a **boundary condition** for the voltage and current:

$$V(z=0) = I(z=0)Z_L \quad \Rightarrow \quad Z_L = \frac{V(0)}{I(0)}$$

- Apply this boundary condition as:

$$V(z=0) = V_o^+ + V_o^-$$

$$I(z=0) = \frac{1}{Z_0}\left(V_o^+ - V_o^-\right)$$




- Hence

$$\frac{V(0)}{I(0)} = \frac{V_o^+ + V_o^-}{V_o^+ - V_o^-} Z_0 = Z_L$$

- Solving for $V_o^- / V_o^+$, and defining this ratio as the **voltage reflection coefficient at the load** $(z=0)$, we find

**complex number**

$$\Gamma_L \equiv \left.\frac{V_o^-}{V_o^+}\right|_{z=0} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

# $Z_{\text{in}}$ of a T-Line-I

$$\because V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$$

$\ell \equiv$ **distance away from load**

$$V(-\ell) = V_o^+ e^{+\gamma\ell} + V_o^- e^{-\gamma\ell}$$

$$= V_0^+ e^{+\gamma\ell}\left(1 + \frac{V_0^-}{V_0^+} e^{-2\gamma\ell}\right)$$

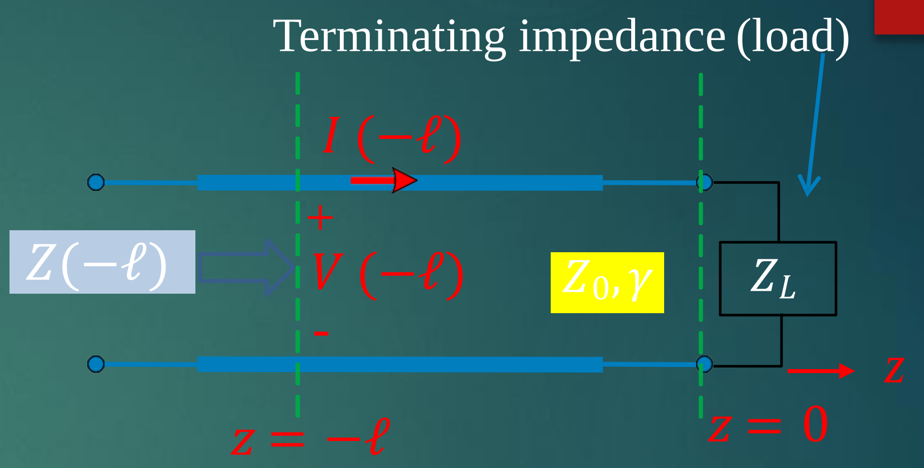
$$= V_0^+ e^{+\gamma\ell}\left(1 + \Gamma_L e^{-2\gamma\ell}\right)$$

- Similarly, the current at $z = -\ell$

$$I(-\ell) = \frac{1}{Z_0}\left[V_o^+ e^{+\gamma\ell} - V_o^- e^{-\gamma\ell}\right]$$

$$= \frac{V_0^+}{Z_0} e^{+\gamma\ell}\left(1 - \frac{V_0^-}{V_0^+} e^{-2\gamma\ell}\right)$$

$$= \frac{V_0^+}{Z_0} e^{+\gamma\ell}\left(1 - \Gamma_L e^{-2\gamma\ell}\right)$$




- Input impedance seen "looking" towards load at $z = -\ell$.

$$Z_{\text{in}} = Z(-\ell) \equiv \frac{V(-\ell)}{I(-\ell)}$$

$$= Z_0 \frac{1 + \Gamma_L e^{-2\gamma\ell}}{1 - \Gamma_L e^{-2\gamma\ell}}$$

$Z(0) = ?$

# $Z_{in}$ of a T-Line-II

- Subsitituting for $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$:
- Thus, $Z(-\ell) = Z_0 \dfrac{1 + \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma\ell}}{1 - \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma\ell}} \times \dfrac{Z_L + Z_0}{Z_L + Z_0}$

$$Z(-\ell) = Z_0 \frac{(Z_L + Z_0) + (Z_L - Z_0)e^{-2\gamma\ell}}{(Z_L + Z_0) - (Z_L - Z_0)e^{-2\gamma\ell}} \times \frac{e^{+\gamma\ell}}{e^{+\gamma\ell}}$$

$$= Z_0 \frac{(Z_L + Z_0)e^{+\gamma\ell} + (Z_L - Z_0)e^{-\gamma\ell}}{(Z_L + Z_0)e^{+\gamma\ell} - (Z_L - Z_0)e^{-\gamma\ell}}$$

$$= Z_0 \frac{Z_L(e^{+\gamma\ell} + e^{-\gamma\ell}) + Z_0(e^{+\gamma\ell} - e^{-\gamma\ell})}{Z_0(e^{+\gamma\ell} + e^{-\gamma\ell}) + Z_L(e^{+\gamma\ell} - e^{-\gamma\ell})}$$

# $Z_{in}$ of a T-Line-III

- Using trigonometric identities:

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right) \quad \text{and} \quad \sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

- Hence, we have

$$Z(-\ell) = Z_0 \frac{Z_L \cosh(\gamma\ell) + Z_0 \sinh(\gamma\ell)}{Z_0 \cosh(\gamma\ell) + Z_L \sinh(\gamma\ell)}$$

- Divide both the numerator and the denominator by $\cosh(\gamma\ell)$

$$Z_{\text{in}} = Z(-\ell) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma\ell)}{Z_0 + Z_L \tanh(\gamma\ell)}$$

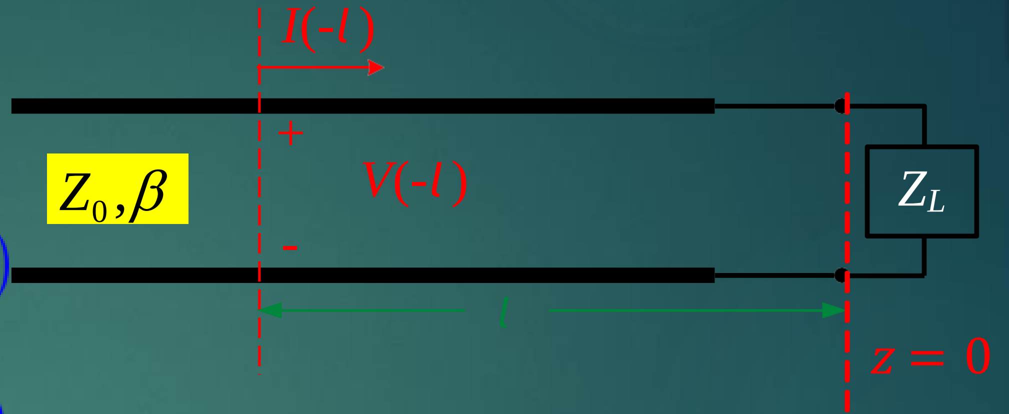
# Input Impedance of a Lossless TL

$$\gamma = \alpha + j\beta = j\beta$$

$$V(-\ell) = V_0^+ e^{j\beta\ell}\left(1 + \Gamma_L e^{-2j\beta\ell}\right)$$

$$I(-\ell) = \frac{V_0^+}{Z_0} e^{j\beta\ell}\left(1 - \Gamma_L e^{-2j\beta\ell}\right)$$

$$Z(-\ell) = Z_0\left(\frac{1 + \Gamma_L e^{-2j\beta\ell}}{1 - \Gamma_L e^{-2j\beta\ell}}\right)$$




$$Z_{\text{in}} = Z(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}$$

tan repeats when

$$\beta\ell = \pi \Rightarrow \frac{2\pi}{\lambda_g}\ell = \pi$$

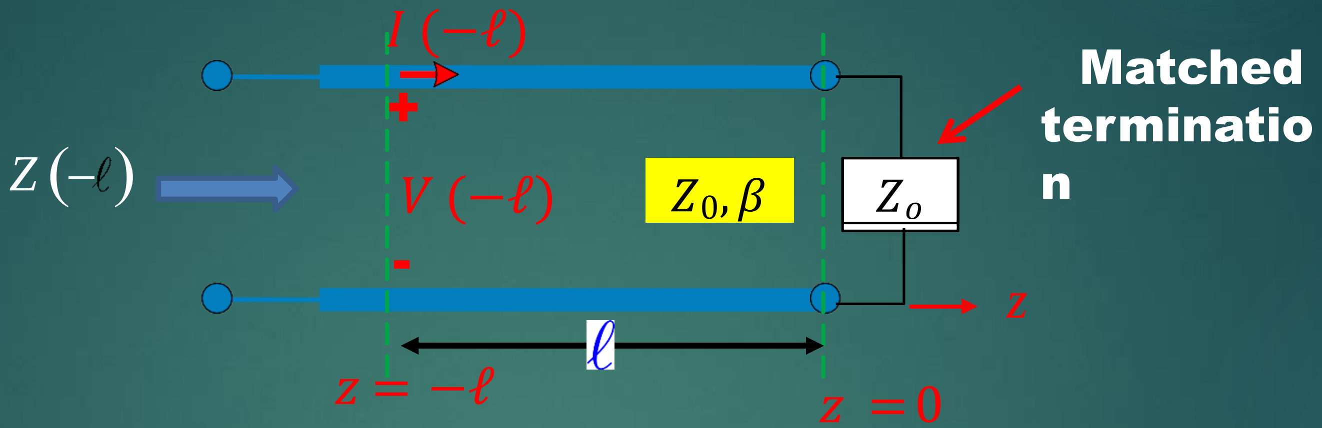
$$\Rightarrow \ell = \lambda_g/2$$

**Impedance is periodic with period $\lambda_g/2$**

**Note:**

$$\tanh(\gamma\ell) = \tanh(j\beta\ell) = j\tan(\beta\ell)$$

# Matched Load ($Z_L = Z_0$)




$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0$$ **No reflection from the load**

$$\Rightarrow V(-\ell) = V_0^+ e^{j\beta\ell}\left(1 + \Gamma_L e^{-2j\beta\ell}\right) = V_0^+ e^{+j\beta\ell}$$

$$I(-\ell) = \frac{V_0^+}{Z_0} e^{j\beta\ell}\left(1 - \Gamma_L e^{-2j\beta\ell}\right) = \frac{V_0^+}{Z_0} e^{+j\beta\ell}$$

$$Z(-\ell) = \frac{V(-\ell)}{I(-\ell)} = Z_0$$

$$Z_{\text{in}} = Z_0 \quad [\Omega]$$

The input impedance is $Z_0$ regardless of the length of the TL.

# Short-Circuit Load ($Z_L = 0$)

$$\Gamma_L = \frac{0 - Z_0}{0 + Z_0} = -1$$

$$\Rightarrow Z(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}$$

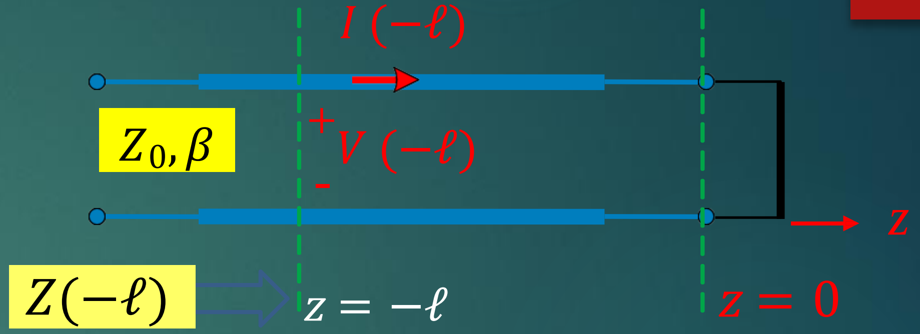
$$= jZ_0 \tan(\beta\ell)$$

**Always imaginary!**

$$\Rightarrow Z(-\ell) = jX_{SC}$$

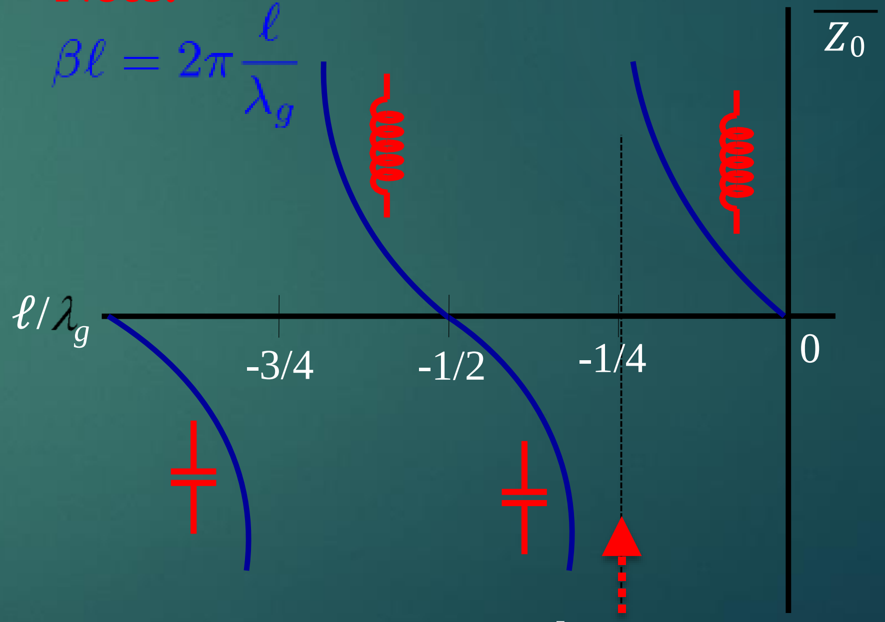
$$\therefore X_{SC} = Z_0 \tan(\beta\ell)$$

$$Z_{in}\Big|_{Z_L=0} = jZ_0 \tan(\beta\ell) \quad [\Omega]$$




**Note:**

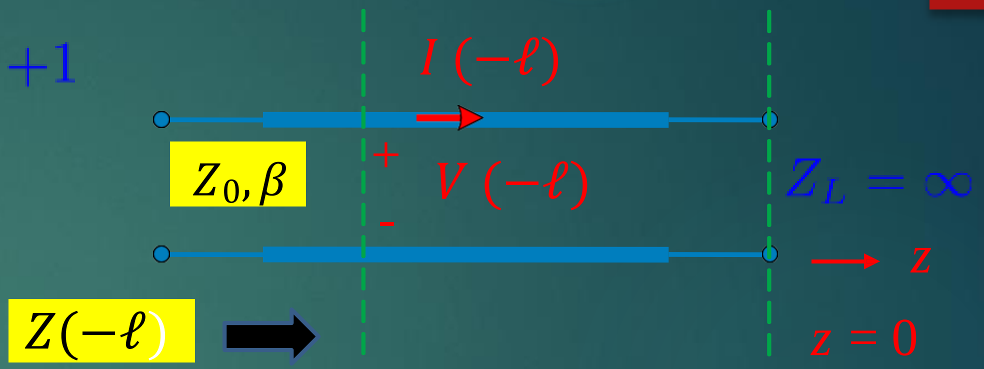
$$\beta\ell = 2\pi \frac{\ell}{\lambda_g}$$




S.C. can become an O.C. with a $\lambda_g/4$ TL

# Open-Circuit Load ($Z_L = \infty$)

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{1 - (Z_0/Z_L)}{1 + (Z_0/Z_L)} = +1$$




$$Z(-\ell) = Z_0 \frac{Z_L + jZ_0 \tan(\beta\ell)}{Z_0 + jZ_L \tan(\beta\ell)}$$

$$= Z_0 \frac{1 + j(Z_0/Z_L)\tan(\beta\ell)}{(Z_0/Z_L) + j\tan(\beta\ell)}$$

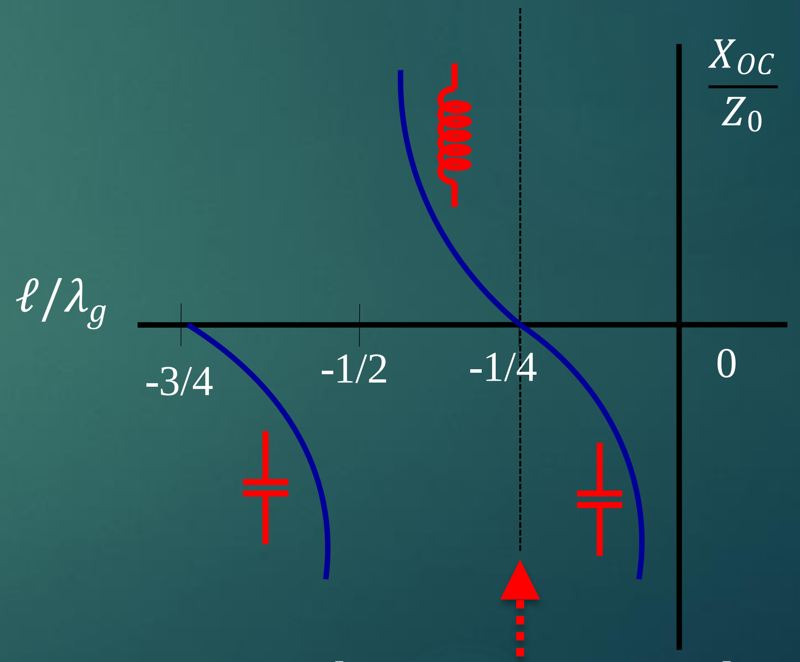
$$= -jZ_0 \cot(\beta\ell)$$

**Always imaginary!**

Note: $Z(-\ell) = jX_{OC}$

$$\Rightarrow X_{OC} = -Z_0 \cot(\beta\ell)$$

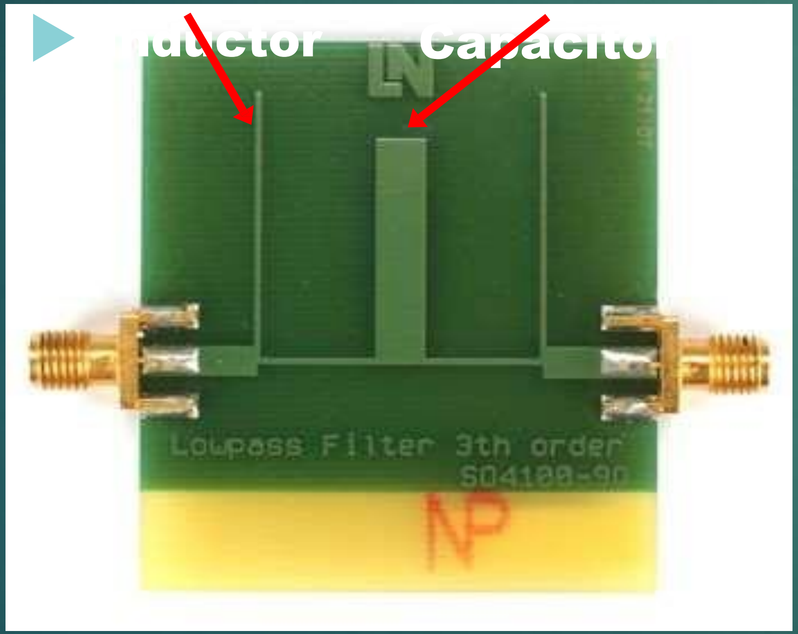
$$Z_{in}\Big|_{Z_L=\infty} = -jZ_0 \cot(\beta\ell) \quad [\Omega]$$

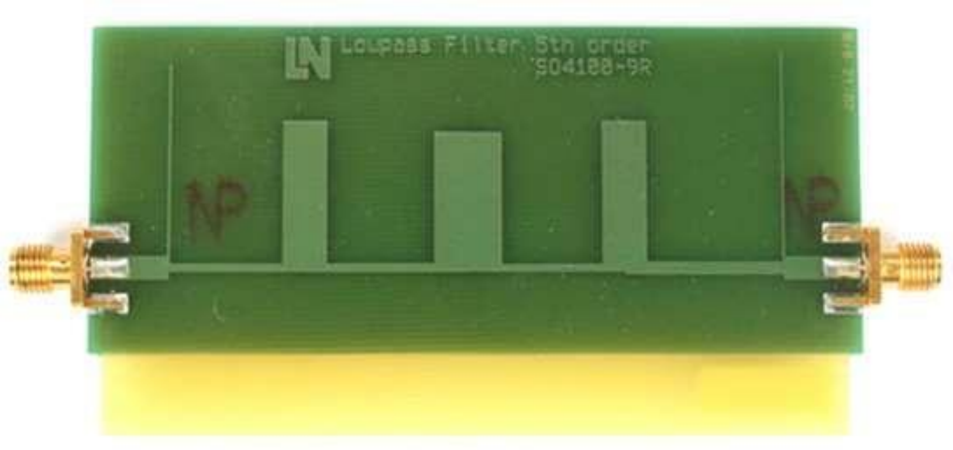



O.C. can become a S.C. with a $\lambda_g/4$ transmission line.

# Using TLs to Synthesize Elements

- We can obtain any reactance that we want from a short or open TL.
- This is very useful is microwave engineering.




A microwave filter constructed from microstrip.

Thank you Very Much !!!