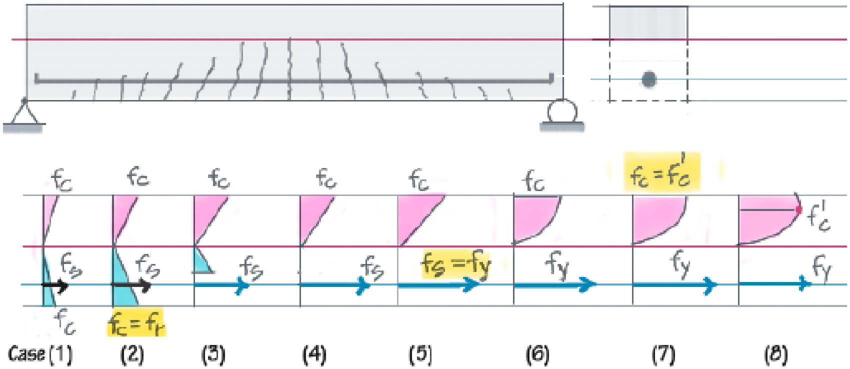
Rectangular Beam Analysis



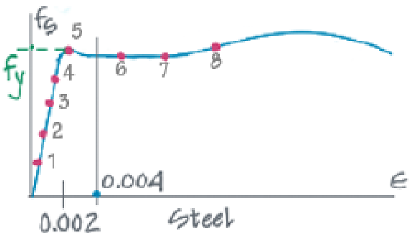


Fig. (a): Stress strain diag. of Steel

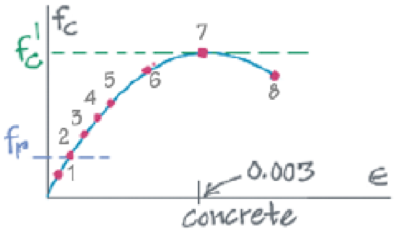
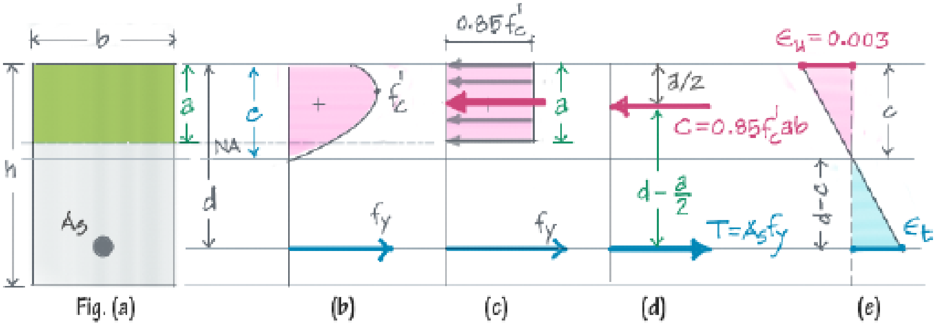


Fig. (b): Stress strain diag. of Concrete

Sym.	Description
f_c	Stress in concrete
f_r	Modulus of rupture of concrete
f'_c	Compressive strength of concrete
f_s	Stress in steel
f_y	Yield strength of steel

- **Case (1):** At small load, both f_y and f_c are smaller. Flexural diagrams of concrete exist in both tensile (cyan) and compressive (magenta) zone. No crack in beam at this stage.
- **Case (2):** As load increases, f_c becomes equal to f_r (Point 2 in Fig.b). Therefore, the first crack appears at midspan.
- **Case (3):** Further increase of load causes the crack to propagate upward. The stress diagram in tensile zone of concrete shrinks.
- **Case (4):** Crack reaches the neutral axis resulting the tensile stress zone of concrete to be completely disappeared.
- **Case (5):** More loads cause f_s to become equal to f_y (Point 5 in Fig.a). Therefore the steel has yielded at this stage near 0.002 strain.
- **Case (6):** Additional load causes concrete strain to enter in nonlinear zone (Point 6 in Fig.b). This results the stress block to become parabolic in shape instead of triangular.
- **Case (7):** Further load forces concrete to reach its maximum stress f'_c (Point 7 in Fig.b) at 0.003 strain. Only the topmost level of concrete experience f'_c.
- **Case (8):** At this ultimate stage, level of f'_c sinks a bit. The concrete reaches its failure stage (Point 8 in Fig.b) and the beam fails by crushing of concrete.

Whitney's Stress Block



- **Fig(b):** The actual stress diagram of concrete (shaded magenta) is somewhat difficult to analyze due to its parabolic shape.
- **Fig(c):** To overcome this, C. S. Whitney proposed to replace the parabolic area by a simple *fictitious rectangular area* which is known as *Whitney's Stress Block*, given that, the center of gravity of both parabolic and rectangular area must coincide. The width of this block is found to be equal to 0.85 f'_c from empirical results and the height (a) is unknown before analysis.

The height of Whitney's stress block is determine by force equilibrium condition of a section. The compressive force resisted by concrete (C) must equal to tensile force resisted by steel (T). Therefore

$$C = T; \qquad 0.85\, f'_c ab = A_s f_y; \qquad \boxed{a = \frac{0.85\, f'_c b}{A_s f_y}}$$

Note that, depth of Whitney's block (a) is slightly smaller than depth of neutral axis (c).

Sym.	Description
b	Width of beam
h	Depth of beam
d	Effective depth of beam
A_s	Cross sectional area of steel
f'_c	Compressive strength of concrete
f_y	Yield strength of steel
c	Depth of neutral axis
a	Depth of Whitney's block
C	Compressive force resisted by concrete
T	Tensile force resisted by steel
ϵ_u	Ultimate strain in conrete
ϵ_t	Strain in tensile reinforcment

Nominal Moment Capacity, M_n

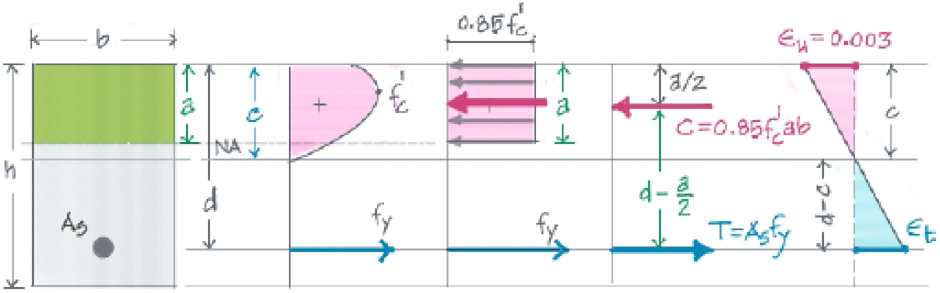


Fig. (a) (b) (c) (d) (e)

- **Fig(d):** The compressive force C is halfway of depth of Whitney's block. Therefore, the distance between C and T becomes $(d - a/2)$.

  To determine the nominal moment capacity of the beam, both compressive force (C) or tensile force T could be used.

  By using Compressive Force:

  Nominal Moment Capacity = Compressive Force × Moment Arm

$$M_n = C\left(d - \frac{a}{2}\right) \qquad \boxed{M_n = 0.85 f'_c ab\left(d - \frac{a}{2}\right)}$$

By using Tensile Force:

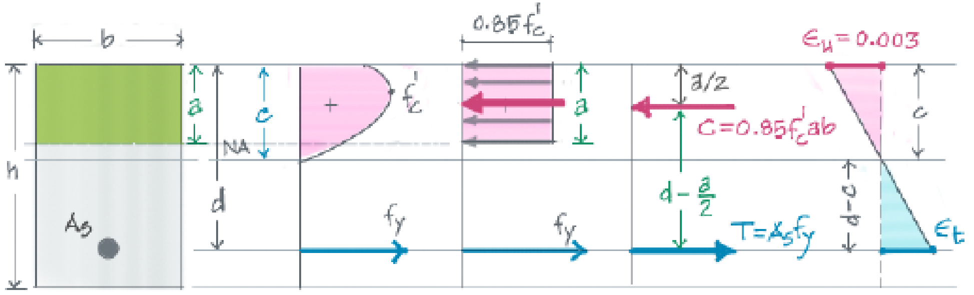
Nominal Moment Capacity = Tensile Force × Moment Arm

$$M_n = T\left(d - \frac{a}{2}\right) \qquad \boxed{M_n = A_s f_y\left(d - \frac{a}{2}\right)}$$

Both formula gives same results but the later one is widely used for determining nominal moment capacity.

Sym.	Description
b	Width of beam
h	Depth of beam
d	Effective depth of beam
A_s	Cross sectional area of steel
f'_c	Compressive strength of concrete
f_y	Yield strength of steel
c	Depth of neutral axis
a	Depth of Whitney's block
C	Compressive force resisted by concrete
T	Tensile force resisted by steel
ϵ_u	Ultimate strain in conrete
ϵ_t	Strain in tensile reinforcment

Strain in Steel, ϵ_t



- **Fig(e):** The ultimate strain of concrete ϵ_u is equal to 0.003, which is found by empirical results.

  Since the depth of neutral axis is c, the distance between steel and neutral axis is equal to $(d-c)$.

  The triangles of Fig.(e), (both cyan and magenta), are similar triangles. Therefore we write,

$$\frac{\epsilon_t}{\epsilon_u} = \frac{d-c}{c}$$

$$\boxed{\epsilon_t = \left(\frac{d-c}{c}\right)\epsilon_u}$$

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b	Width of beam
h	Depth of beam
d	Effective depth of beam
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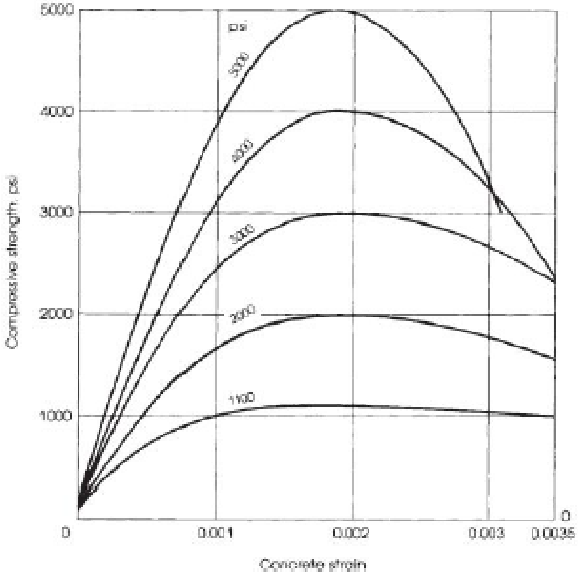
The β_1 and ϕ Factors



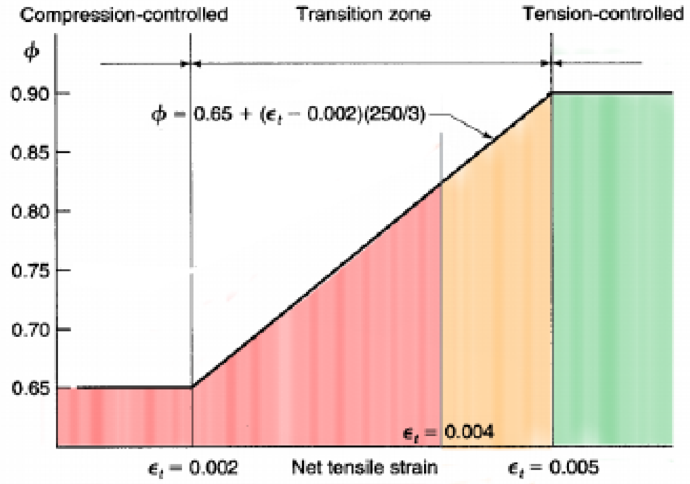
β_1 factor relates **depth of neutral axis** (c) to compressive strength of concrete (f'_c)

If, $f'_c \leq 4000$ psi,

$$\boxed{\beta_1 = 0.85}$$

If, $f'_c > 4000$ psi, the following formula should be used, but the resulting β_1 must be greater than or equal 0.65

$$\boxed{\beta_1 = 0.85 - 0.05\left(\frac{f'_c - 4000}{1000}\right) \geq 0.65}$$



ϕ factor accounts **how far steel has yielded**.

If, $\epsilon_t \geq 0.005$,

$$\boxed{\phi = 0.90}$$

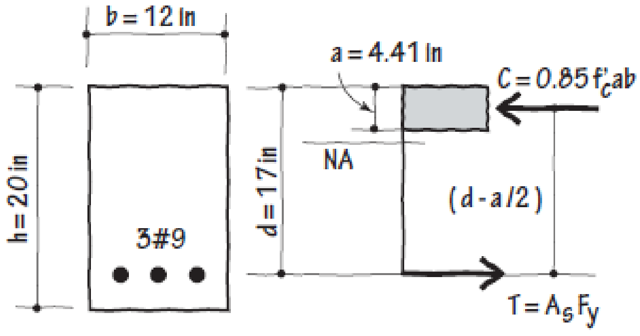
If, $0.004 \leq \epsilon_t \leq 0.005$,

$$\boxed{\phi = 0.65 + \frac{250}{3}(\epsilon_t - 0.002)}$$

Note: The above formula is valid only for Grade 60 steel ($f_y = 60$ ksi).

ACI code does not permit $\epsilon_t \leq 0.004$ for beam, that zone is only reserved for column.

Ex. 1, Nominal Capacity



Ques. Determine the nominal moment capacity of the beam. Given that, $f'_c = 4$ ksi, $f_y = 60$ ksi.

Solution.

1. Set compression equals tension to find depth of Whitney's stress block.

$$C = T$$

$$0.85\, f'_c ab = A_s f_y$$

$$a = \frac{A_s f_y}{0.85 f'_c b}$$

$$= \frac{3.00 \times 60}{0.85 \times 4 \times 12}$$

$$= 4.41 \text{ in}$$

2. Determine nominal moment that the beam can carry.

$$M_n = A_s f_y \left(d - \frac{a}{2}\right)$$

$$= 3.00 \times 60 \left(17 - \frac{4.41}{2}\right)$$

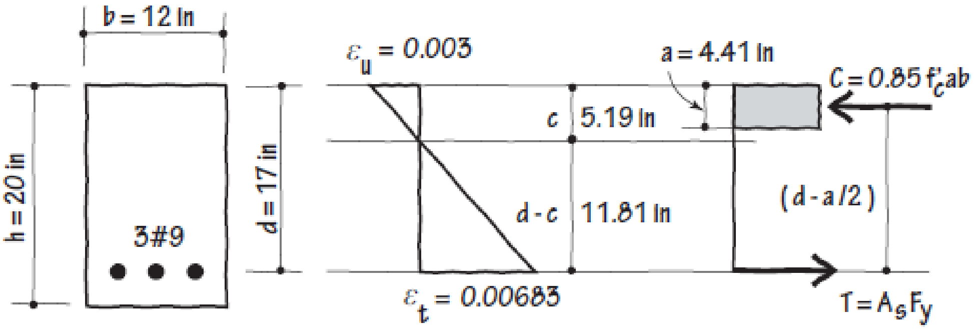
$$= 2663 \text{ k-in}$$

$$= 221.9 \text{ k-ft}$$

Ans. 221.8 kft

Ex. 2, Design Capacity when $\epsilon_t > 0.005$

Ques. Determine the ultimate moment capacity of the following beam. Given that, $f'_c = 4$ ksi, $f_y = 60$ ksi.



Solution.

1. Find depth of Whitney's stress block.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3.00 \times 60}{0.85 \times 4 \times 12} = 4.41 \text{ in}$$

2. Since $f'_c \leq 4000$ psi, $\beta_1 = 0.85$.

3. Find location of neutral axis.

$$c = \frac{a}{\beta_1} = \frac{4.41}{0.85} = 5.19 \text{ in}$$

4. Check if tensile steel has yielded beyond strain of 0.005.

$$\epsilon_t = \left(\frac{d - c}{c}\right)\epsilon_u = \left(\frac{17 - 5.19}{5.19}\right)0.003 = 0.00683$$

5. Since $\epsilon_t \geq 0.005$, $\phi = 0.90$.

6. Determine ultimate moment capacity.

$$\phi M_n = \phi A_s F_y \left(d - \frac{a}{2}\right)$$

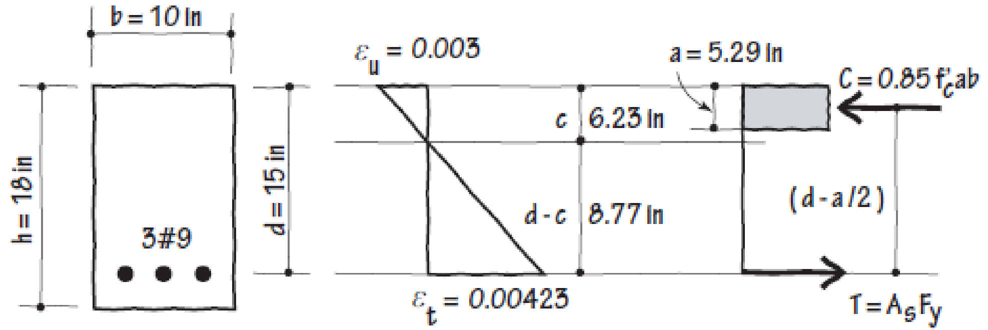
$$= 0.90 \times 3.00 \times 60 \left(17 - \frac{4.41}{2}\right)$$

$$= 2396 \text{ k-in} = 199.7 \text{ k-ft}$$

Ans. 199.7 kft

Ex. 3, Design Capacity when $0.004 < \epsilon_t < 0.005$

Ques. Determine the ultimate strength of the section. Given that, $f'_c = 4$ ksi, $f_y = 60$ ksi.



Solution.

1. Find depth of stress block.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3.00\times 60}{0.85\times 4\times 10} = 5.29 \text{ in}$$

2. Since $f'_c \leq 4000$ psi, $\beta_1 = 0.85$.
3. Find location of neutral axis.

$$c = \frac{a}{\beta_1} = \frac{5.29}{0.85} = 6.23 \text{ in}$$

4. Check if tensile steel has yielded beyond strain of 0.005.

$$\epsilon_t = \left(\frac{d-c}{c}\right)\epsilon_u = \left(\frac{15-6.23}{6.23}\right)0.003 = 0.00423$$

5. Since $\epsilon_t < 0.005$, ϕ does not equal to 0.90 anymore.

$$\phi = 0.65 + (\epsilon_t - 0.002)\frac{250}{3}$$

$$= 0.65 + (0.00423 - 0.002)\frac{250}{3} = 0.836$$

6. Determine ultimate moment.

$$\phi M_n = \phi A_s F_y \left(d - \frac{a}{2}\right)$$

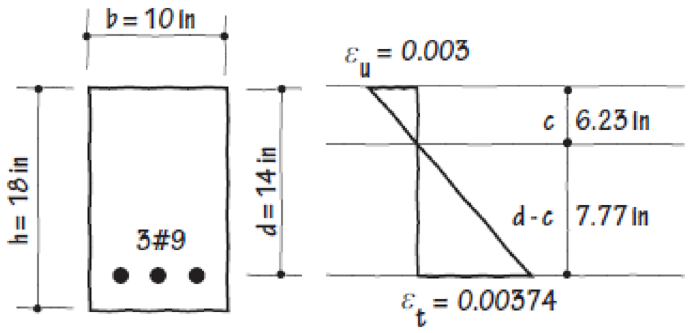
$$= 0.834\times 3.00\times 60\left(15 - \frac{5.29}{2}\right)$$

$$= 1859 \text{ k-in} = 154.9 \text{ k-ft}$$

Ans. 154.9 kft

Ex. 4, Design Capacity when $\epsilon_t < 0.004$

Ultimate Strength of Rectangular Beam



Ques. Determine the ultimate moment capacity of the following beam. Given that, $f'_c = 4$ ksi, $f_y = 60$ ksi.

Solution.

1. Find depth of stress block.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3.00 \times 60}{0.85 \times 4 \times 10} = 5.29 \text{ in}$$

2. Since $f'_c \leq 4000$ psi, $\beta_1 = 0.85$.

3. Find location of neutral axis.

$$c = \frac{a}{\beta_1} = \frac{5.29}{0.85} = 6.23 \text{ in}$$

4. Check if tensile steel has yielded beyond strain of 0.005.

$$\epsilon_t = \left(\frac{d - c}{c}\right)\epsilon_u = \left(\frac{14 - 6.23}{6.23}\right)0.003 = 0.00374$$

Since $\epsilon_t < 0.004$, section is not ductile. Therefore, the beam **cannot be used**. The size of the beam (specifically depth) should be increased or the amount of reinforcement should be decreased so that $\epsilon \geq 0.005$ becomes true.

Ans. Increase beam size or reduce reinforcement.